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The Gift of

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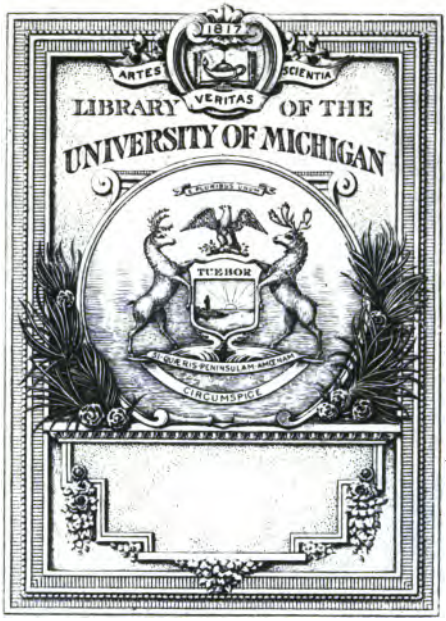
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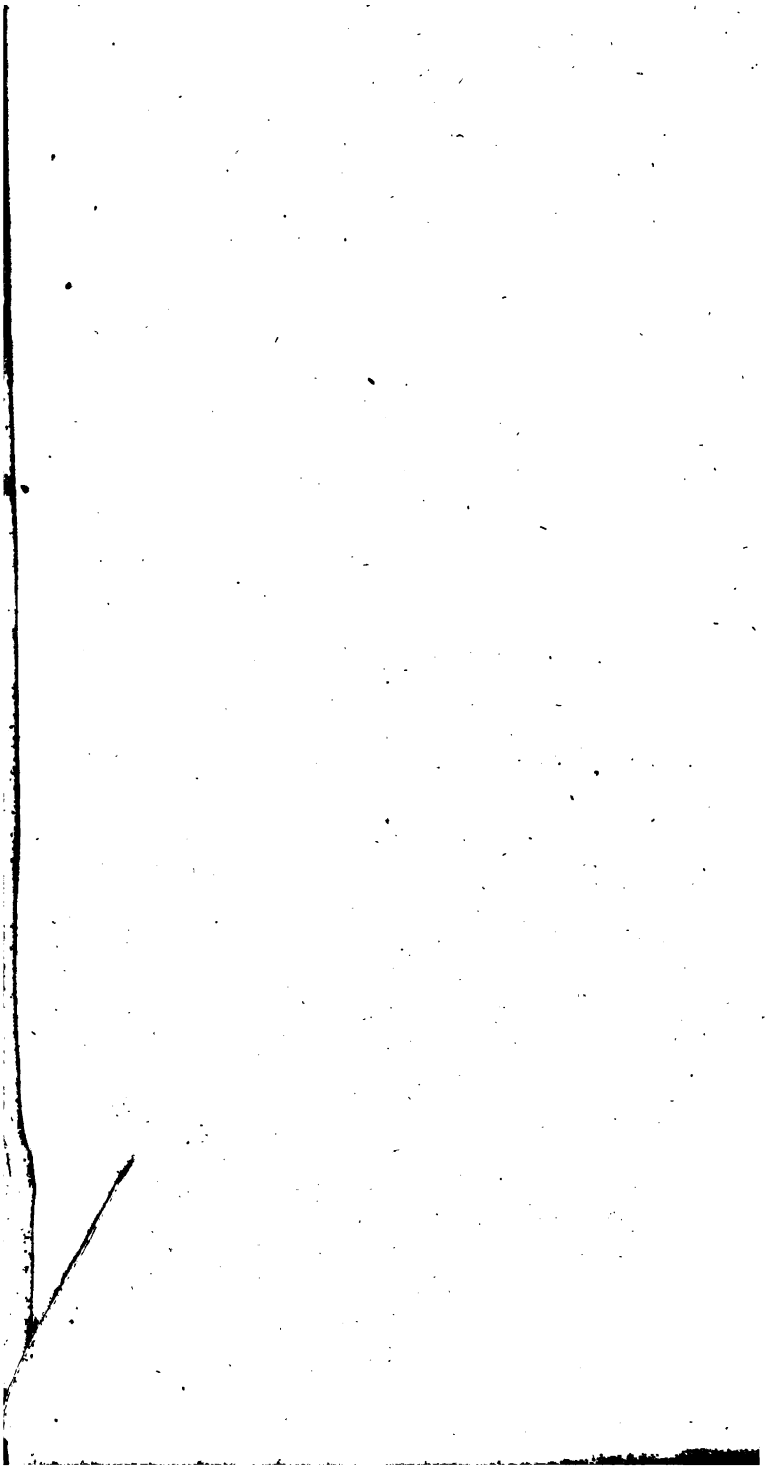

MISCELLANIES.

OR A

MISCELLANEOUS TREATISE,

CONTAINING

SEVERAL MATHEMATICAL

SUBJECTS.

RADCLIFFE OBSERVATORY OXFORD.

——— *Non ego paucis*
Offendor Maculis, ———

HOR.

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Gift
Professor William H. Pettis
10-14-1930

PREFACE.

THE following Book conſiſts of ſeveral Articles, in divers parts of the Mathematics. 1. *The* Laws of Chance; *the principles of which are delivered in a ſhort method, and is preparatory to what follows.* 2. *Of finding the* values of Annuities on Lives; *here the principal Problems concerning annuities and reverſions are ſolved.* 3. *The* inſtituting or forming Societies; *a thing much in vogue now. Here ſeveral of the caſes are calculated upon true principles; by which it will appear, that hardly any of theſe Societies now in uſe can ſtand for any length of time upon the principles they have been calculated. In theſe three Articles, to avoid long calculations, I have frequently uſed approximations, which in moſt caſes may ſerve well enough, for ſuch an inconſtant ſubject as Chance.* 4. *Of* the Moon's Motion, *where ſeveral of her irregularities are calculated.* 5. The Conſtruction of Arches; *here are laid down ſeveral methods of performing this.* 6. The Preceſſion of the Equinoxes; *which is univerſally ſolved upon all ſuppoſitions.* 7. The Conſtruction of Logarithms, *by the roots or indices of powers.* 8. Interpolation of Quantities, *the beſt methods of doing this.* 9. *Of finding the* Longitude at Sea; *this method is the ſame as that which I gave a ſhort account of in page* 364, *of my Aſtronomy; which took up but a ſingle page; and yet I got myſelf abuſed for it, by an anonimous ſcribbler. I never knew before, that it was any crime to ſeek for new methods for ſolving difficult Problems. If I had failed in it, there was but one page of the Book loſt. But here I have improved the method, and compleated*

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it; and have ſhewn, that it will anſwer the purpoſe as near as there can be any occaſion for. Indeed, I have conſtructed no Tables to eaſe the calculation. For though it requires a little more time, I ſhould ſooner chuſe to go along the common high road, rather than go over hedge and ditch for the ſake of a little gain. And this method I deſign only to be uſed occaſionally; as the common known methods are ſufficient in moderate weather. For certainly no body can be ſo ſilly as to think, that any methods by the Moon ought to be practiſed every day. Theſe that are reckoned the beſt methods, have ſo many calculations attending them, that there is infinite danger of making miſtakes, and therefore ought to be uſed ſparingly.

10. *The computation of* Intereſt *upon all manner of ſuppoſitions.* 11. *The* Figures of Sines, Tangents, *&c. and ſeveral things depending thereon.* 12. Fortification, *a ſhort tract, laying down the general rules.* 13. Gunnery, *ſhewing the principles of that art.* 14. Architecture, *or the Art of Building, containing the rules and practice of this art.* 15. Muſic, *the nature and principles laid down.* 16. *A* general Method of Philoſophiſing, *or rules to be obſerved in Philoſophical Enquiries, with Examples.* 17. Optics; *I have here tranſlated ſeveral papers of* Sir J. Newton's Optical Lectures, *Part* II. *which never was tranſlated into Engliſh. The Phænomenon of the Rainbow is here computed in a way ſomething different from what it is in his* Optics, *being here illuſtrated and wrought out in numbers, and therefore will be more intelligible. I wonder this latter part of the Optical Lectures has never been tranſlated. For tho' the ſubſtance of it is contained in the Optics, yet here are a great number of curious experiments not mentioned there, which deſerve to be made public.* 18. A collection of curious, uſeful, and important Problems, *many of them new; and thoſe that are not, have new Solutions.*

I have

I have done all I could to keep clear of errors, but among ſo many things, ſome may poſſibly eſcape, for I have no pretence to infallibility; and if my Readers will not excuſe me in this, they muſt not read my Book.

But I cannot diſmiſs this affair, without taking notice of a certain obſcure Critic, who inſulted and abuſed me in the Gentleman's Magazine, for ſome trifling numerical errors in the Aſtronomy, that any body, with the leaſt ſmattering of Mathematics, might rectify. Yet this Drawcanſir *kept barking at me out of his hole, in ſeveral of the Magazines, and pretended to criticize my Book, though he acknowledged he did not underſtand it. A fine ſort of a Critic indeed, to rail at me for his own dulneſs. Yet this doughty Author, when he came to be unkennelled, proved to be no more than a little paltry Schoolmaſter, who did not ſo much as underſtand his own native Language; nor (by report) ſcarce ever writ a line of Mathematics in his life. This mighty Hero, in the laſt paper (which his truſty friend, the Compiler of the Magazine, put in for him) has inſerted no leſs than a ſcore if falſhoods, either directly ſuch, or ſtrongly inſinuated. Yet this ſame impartial Compiler refuſed to do me the juſtice to put in my Remarks, which expoſed theſe enormous lies, becauſe his Friend and Favourite was going to give up. So that my paper is ſtill in the hands of Mr.* Nourſe, *where any body may ſee it that pleaſes.*

Therefore my Readers may pleaſe to take notice, that if any envious, abuſive, dirty Scribbler, ſhall hereafter take it in his head to creep into a hole like an Aſſaſſin, and lie lurking there on purpoſe to ſcandalize and rail at me; and dare not ſhew his face like a Man; I ſhall give myſelf no manner of trouble about ſuch an Animal, but look upon him as even below contempt.

W. EMERSON.

POST-

POSTSCRIPT.

Since Mayer's Tables came into my hands, I had a mind to try their exactneſs, by comparing them with Dr. Halley's Obſervations. Accordingly I found them to come very near in moſt caſes, but not in all. For in many caſes Dr. Halley's Tables come nearer; and ſo do ſome other Tables that I tried. And in making theſe compariſons here and there, I found ſometimes an error of 2 minutes or more, and ſeveral not much ſhort of 2 minutes. For example, Dec. 31ſt, 1725, the error is — 2′ 45″. *Jan. 31ſt, 1726, the error is* +2′ 21″. *And Apr. 15th, 1728, the error is* + 2′ 3″. *So that it may be truly ſaid, that* no Tables extant *can be depended upon to give the Moon's place true to two minutes; contrary to the confident aſſertions of our little* A, B, C, *Critic, who has been ſo hot upon the matter.*

And here I call upon this fiery Zealot, this boaſting, upſtart, nominal Aſtronomer, that among the many ſtrange poſitions he has laid down, he will explain to the public the following particulars.

1. *How he can make it intelligible, or even ſhew its poſſibility; that in taking the diſtance of two objects, it is only neceſſary to obſerve or look at one of them.*

2. *How he can make it appear,* that the depriving a luminous body of its light, is an abſurdity.

3. *That he will point out* theſe Corollaries (being near half in the Book *of Aſtronomy*) *that have no dependance upon the Propoſitions they are placed under.*

4. *How he makes it out, that I have* calumniated and condemned Mayer's Works; *and* wilfully and wickedly calumniated the Nautical Almanack. *And particularly to ſhew what actions I have committed, which he ſays are* worſe than robbing the Widow and Fatherleſs.

5. *How*

5. *How it comes to be* a fundamental principle in Philoſophy, *to aſcribe the properties of a ſolid body to a mere image.*

6. *That, for the public good, and to keep men out of error, he will acquaint them with, and expoſe, the ſeveral abſurdities he has lately found, and is always finding in my Book.*

7. *To aſſign a reaſon why the Eclipſes of Jupiter's Satellites are not viſible* till long after *the time expected, perhaps* a quarter of an hour.

8. *By what authority he perverts the meaning of my words, and impoſes his own ſenſe upon them, directly contrary to mine.*

This aſtronomical Conjuror will eaſily find his way into the Magazine; and when he has fully explained and cleared up all theſe Articles, men will be able to form a judgment of his mathematical abilities, as well as of his moral character, and, doubtleſs, the impartial world will judge accordingly.

W. E.

CON-

CONTENTS.

ART. I. Laws of Chance. — p. 1
ART. II. Annuities. — — 49
ART. III. Societies. — — 121
ART. IV. Moon's Motion. — — 139
ART. V. Conſtruction of Arches. — 148
ART. VI. Preceſſion of the Equinoxes. 180
ART. VII. Conſtruction of Logarithms. 189
ART. VIII. Interpolation. — — 199
ART. IX. The Longitude. — — 204
ART. X. Intereſt. — — 226
ART. XI. Figure of Sines, &c. — 232
ART. XII. Fortification. — — 242
ART. XIII. Gunnery. — — 277
ART. XIV. Architecture. — — 322
ART. XV. Muſic. — — 359
ART. XVI. Rules of Philoſophy. — 405
ART. XVII. Optical Lectures. — 413
ART. XVIII. Problems. — — 428

ART.

ART. I.

THE LAWS OF CHANCE.

THOUGH Chance may ſeem a fickle and ludicrous ſubject to treat upon, becauſe there is no certainty or conſtancy in it. Yet all the events that happen have ſome certain degrees of probability for their happening or not happening; and theſe degrees of probability come under a mathematical conſideration. Although it is impoſſible to determine with certainty how an event ſhall happen, yet it may be determined mathematical, what likelihood or degree of probability there is for its happening or failing; and this is all that is intended by the calculation. For ſcarce any thing falls out as it is determined by calculation, except there be made an infinite number of repetitions, and then one with another will always bring it to the ſame thing as the calculation makes it.

The Doctrine of Chances, therefore will be very uſeful to ſuch perſons as engage in play; for by this means they may know what advantage or diſadvantage they have on their ſides; and how to manage things to the beſt advantage in the play.

It is likewiſe a fit ſubject for the exerciſe of reaſon, for many problems about Chance; though they ſeem to be very ſimple, are attended with a long train of conſequences, before we come at the

concluſion, which is often quite different from what one would expect. And thus it corrects ſuch miſtakes as we are apt to make in our judgment.

DEFINITION I.

Chance is an event, or ſomething that happens without the deſign or direction of any agent; and is directed or brought about by nothing but the laws of nature.

Therefore Chance has a natural, but not a final cauſe. It is the very nature of Chance to be inconſtant.

DEF. II.

The *probability* or *improbability* of an event happening, is the judgment we form of it, by comparing the number of Chances there are for its happening, with the number of Chances for its failing.

DEF. III.

Expectation in play, is the value of a man's Chance; that is, of the thing played for, conſidered with the probability of gaining it; and therefore is the product of its value multiplied by the probability of obtaining the prize.

And the *advantage* = *expectation* — the *ſtake*.

DEF. IV.

Riſk is the value of the ſtake conſidered with the probability of loſing it; and therefore is the product of its value multiplied by the probability of loſing it.

DEF. V.

Events are *independent* when they have no manner of connection with one another; or when the happening of one neither forwards nor obſtructs the happening of any other of them.

Thus

Thus if A undertake to throw an ace at two throws with one die; the firſt throw he has 1 Chance to win and 5 to loſe; and the ſecond throw he has alſo the very ſame, viz. 1 chance to win and 5 to loſe; and the like as often as he pleaſes, and therefore theſe events are independent.

DEF. VI.

An event is *dependent* when the probability of its happening is altered by the happening of ſome other.

Thus if A undertakes to draw the ace out of 13 cards of a ſuit, at two trials or more. At the firſt trial he has 1 Chance to win and 12 to loſe; at the ſecond trial, a card being now taken away, he has 1 Chance to win and 11 to loſe; and at the third, 1 Chance to win, and 10 to loſe, &c. therefore theſe events are dependent.

AXIOM I.

In computing the number of Chances, it is ſuppoſed that all Chances are equal, or made with equal facility.

AXIOM II.

The whole expectation for any prize, is the ſum of all the expectations upon the particulars.

AXIOM III.

The value of any Chance or expectation is what would purchaſe the like Chance or expectation, in a fair game.

PROP. I.

Putting 1 *for abſolute certainty, a for the number of caſes (or chances) for the happening of an event.*

b for the number of caſes for its failing. Then $\frac{a}{a+b}$ *is the probability of its happening; and* $\frac{b}{a+b}$ *is the probability of its failing: and the probabilities of happening and failing are as a to b.*

For ſuppoſe a and b to be equal; no reaſon can be given why ſuch an event will not as often fail as happen. And therefore the probability will be equal for both. And if a be greater or leſſer than b, the probability will accordingly be greater or leſs for happening than for failing, and that in the ratio of a to b, or of $\frac{a}{a+b}$ to $\frac{b}{a+b}$.

Cor. *The probability ſubtracted from* 1, *gives the probability of failing.*

For $1-\frac{a}{a+b}=\frac{a+b-a}{a+b}=\frac{b}{a+b}$ the probability of failing. And here 1 repreſents the ſum of all the caſes or chances, for and againſt happening. For it is a certainty, that ſome out of the whole muſt happen.

EXAMPLE.

If I undertake to throw an ace with one die; it is plain I have but 1 chance out of 6 for its happening; and 5 out of 6 for its failing; therefore my chance is $\frac{1}{1+5}$ or $\frac{1}{6}$, being the probability of its happening; and $\frac{5}{6}$ is the probability of its failing.

SCHOLIUM.

As 1 repreſents a certainty, or when an event has an infinitely great probability of happening. So $\frac{1}{2}$ will repreſent an equal probability for happening or failing. For $1-\frac{1}{2}=\frac{1}{2}$.

PROP.

PROP. II.

If a be the number of chances for happening, and b the number of chances for the failing, of one event.

And c the number of chances for happening, and d the number of chances for failing of another event independent of the former. Then

The product $\overline{a+b} \times \overline{c+d}$ *or* $ac+ad+bc+bd$ *will be the number of all the chances, or of all the poſſible caſes of the happening and failing of theſe events.*

For $a+b$ denotes all the chances in the firſt event, and ſince $a+b$ may be connected with every one in $c+d$, therefore the ſum of all the caſes will be $\overline{a+b} \times \overline{c+d}$.

Cor. 1. *By the ſame reaſoning, if a, c, e, &c. be the chances for happening; and b, d, f, &c. thoſe for failing, in 3 or more independent events. Then* $\overline{a+b} \times \overline{c+d} \times \overline{e+f}$, *&c. will be the number of all the chances or poſſible caſes for all theſe events, to happen or fail.*

Cor. 2. *If there be propoſed two events, the number of caſes for the happening of the firſt, is* $ac+ad$; *for its failing,* $bc+bd$. *That both ſhall happen, is ac; for the contrary,* $ad+bc+bd$. *That one ſhall happen; is* $ac+ad+bc$; *for both failing, bd; that the firſt ſhall happen and the latter fail, is* ad; *for the contrary,* $ac+bc+bd$. *and ſo of others.*

Cor. 3. *And by the ſame way of reaſoning, if three or more events be propoſed; the number of caſes of their happening or failing, in any manner, will be had by multiplication only. So that by taking one or more terms according to the nature of the queſtion, any problem about chance will eaſily be anſwered.*

PROP. III.

The probability of happening of ſeveral independent events $\left(\frac{a}{a+b}, \frac{c}{c+d}, \frac{e}{e+f}, \&c.\right)$ *is the product of the ſeveral probabilities taken ſingly,* $\frac{ace}{a+b \times c+d \times e+f}$.

For in one event, $\frac{a}{a+b}$ is the probability of happening, by Prop. I. and in two events, where the probabilities are $\frac{a}{a+b}$ and $\frac{c}{c+d}$; $\overline{a+b} \times \overline{c+d}$ is the number of all the caſes or chances, and *ac* the number of caſes where both events may happen; therefore $\frac{ac}{\overline{a+b} \times \overline{c+d}}$ = probability of both happening. In like manner $\frac{ace}{\overline{a+b} \times \overline{c+d} \times \overline{e+f}}$ = probability of three events happening. And the Prop. holds in reſpect of failing, as well as of happening of events.

Cor. 1. *If* x, y, z, &c. *denote the probabilities of ſeveral events happening as* A, B, C, &c. *then* $1-x$, $1-y$, $1-z$, &c. *will be the probabilities of their failing. Then*

xyz = *probability of their all happening.*

$\overline{1-x} \times \overline{1-y} \times \overline{1-z}$ = *probability of their all failing.*

$1 - \overline{1-x} \times \overline{1-y} \times \overline{1-z}$ = *probability of ſome of them happening.*

$xy \times \overline{1-z}$ = *probability of happening of* A, B, *and failing of* C.

And ſo of others.

Cor.

Cor. 2. *What is ſaid about the probability of happening, is equally true about the probability of failing, viz. that it is equal to the product of the probabilities of failing of the ſeveral particulars.*

Cor. 3. *To determine the probability of happening of ſeveral dependent events; diſtinguiſh them into firſt, ſecond, third, &c. Then the probability of the firſt muſt be looked upon as independent, the probability of the ſecond's happening, muſt be determined on the ſuppoſition of the firſt having happened ſome way or other; and the probability of the third's happening, is to be determined from the ſuppoſition of the firſt and ſecond having likewiſe happened; and ſo on. And then the probability of the happening of them all will be the product of the ſeveral probabilities as thus determined.*

PROP. IV.

If a be the number of chances for happening, and b for the failing of an event; the probability of happening at leaſt t times in n trials will be

$$= \frac{a^n + na^{n-1}b + \frac{n.n-1}{2}a^{n-2}b^2 + \&c. \text{ till } a^t}{a+b^n}$$

that is, till the index of a be t.

For by Prop. II. and its corollaries, if a, b, and c, d, and e, f, &c. be the chances for happening and failing of 3 or more events A, C, E; then the terms (of the multiplication) where a is concerned, will be the numerator of a fraction, ſhewing the probability of the firſt event happening. And the terms where a and c are concerned, give the numerator of a fraction where the firſt and ſecond events both happen, and ſo on; and whoſe denominator is $\overline{a+b} \times \overline{c+d} \times$ &c.

Now let $c+d$, $e+f$, &c. be the ſame event as $a+b$, and then $\overline{a+b}\times\overline{c+d}\times\overline{e+f}$, &c. $=\overline{a+b}\times\overline{a+b}\times\overline{a+b}$ &c. $=\overline{a+b}^{n}$; and c, e, &c. being $=a$; the happening of a and c is aquivalent to the happening of a twice, and the happening of a, c, e; is equivalent to the happening of a thrice, and ſo on. And therefore the terms with a relate to the happening of the event once, and where aa is concerned, relate to happening twice; and a^3 to happening thrice. Therefore all terms with a as far as a^t, muſt be taken into the numerator for happening t times, and the denominator will be $\overline{a+b}^{n}$; and the fraction, ſhewing the probability of happening t times, will be that above laid down.

Cor. 1. *If a and b be the chances for happening and failing of an event; then* $1-\frac{b^n}{\overline{a+b}^{n}}$ *is the probability of its happening, and* $\frac{b^n}{\overline{a+b}^{n}}$ *the probability of its failing once in n trials.*

For b^n is the only term in which a is not found, and therefore $\frac{b^n}{\overline{a+b}^{n}}$ is the probability of its failing, which taken from 1 gives the probability of happening.

Cor. 2. *The probability of happening preciſely t times in n trials is* $\frac{T}{\overline{a+b}^{n}}$, *where* T *is that term, where the index of a is t.*

PROP. V.

If a perſon has an equal chance for the two prizes A, B; *the value of his expectation is* $\frac{A+B}{2}$.

For

For ſince there is no reaſon for one chance happening ſooner than another, in an equality of chances; therefore (by Prop. I.) the probability of the happening of each of the two is $\frac{1}{2}$; and therefore the expectation for A is $\frac{1}{2}$ A; and the expectation for B is likewiſe $\frac{1}{2}$ B. And if I would purchaſe the whole, I muſt give the value, which is $\frac{1}{2} \times \overline{A+B}$.

For the ſame reaſon, the probability of taking any one of 3 things A, B, C, (by equality of chances) is $\frac{1}{3}$ (by Prop. I); whence the expectation of A is $\frac{1}{3}$ A, and of B, is $\frac{1}{3}$ B, and of C, $\frac{1}{3}$ C, and the whole value $= \frac{1}{3} \times \overline{A+B+C}$, whence this

Cor. *If there be an equal chance for 3 prizes* A, B, C; *the expectation* $= \frac{A+B+C}{3}$. *And for* 4 *things* A, B, C, D; *the expectation is* $\frac{A+B+C+D}{4}$; *and ſo on.*

Schol. The two laſt propoſitions are very ſerviceable for ſolving many problems of Chance.

PROP. VI.

If there be a chance for the prize A, *and b chances for the prize* B; *and all the chances happen with equal eaſe; then the value of the expectation is worth* $\frac{aA+bB}{a+b}$

For (by Prop. I.) the probability of the happening of any chance for A is $\frac{a}{a+b}$; and for B, is $\frac{b}{a+b}$. Therefore the expectation for A is $\frac{aA}{a+b}$; and

and for B is $\frac{bB}{a+b}$. Therefore if I would purchaſe theſe expectations, I muſt give the value of the whole, (by Ax. 2.) which is $\frac{aA+bB}{a+b}$.

And by the ſame reaſoning, if there be 3 prizes A, B, C; and the chances for each be a, b, c, reſpectively. Then the probability for A is $\frac{a}{a+b+c}$; and for B, $\frac{b}{a+b+c}$; and for C, $\frac{c}{a+b+c}$. And the expectation for A $= \frac{aA}{a+b+c}$, for B $= \frac{bB}{a+b+c}$, for C $= \frac{cC}{a+b+c}$. Whence this

Cor. *If a, b, c, be the chances for* A, B, C, *reſpectively; the value of the expectation* $= \frac{aA+bB+cC}{a+b+c}$; *and ſo on.*

PROP. VII.

If a and b be two numbers of particular caſes relating to A *and* B. *And p be ſome number of chances, alike related to a and b. Then the ſhare of p belonging to* A *is* $\frac{a}{a+b}p$; *and* B's *ſhare* $\frac{b}{a+b}p$.

For ſince p is promiſcuouſly and alike related to the whole a and b; it muſt be divided between them, in proportion to the numbers a and b. And the greater number muſt have the greater ſhare, and the leſſer number the leſſer. Therefore if $a+b=s$, it will be as $s : p :: a : \frac{pa}{s}$ = A's ſhare; and $s : p :: b : \frac{pb}{s}$ = B's ſhare.

Cor.

Cor. *If there be ſeveral ſubjects* A, B, C, D, &c. *whoſe numbers are* a, b, c, d, &c. *And the number of chances or things* p *be alike related to them all. Then the ſhares of* A, B, C, D, &c. *are reſpectively* $\frac{ap}{s}$, $\frac{bp}{s}$, $\frac{cp}{s}$, $\frac{dp}{s}$, &c.

SCHOL.

What is meant here by A, B, C, D, &c. is any ſpecies of beings; and by a, b, c, d, the like ſpecies of beings under particular circumſtances; and which may belong equally to any of them.

PROP. VIII.

If there be n *equal chances in all, and* 1 *for an event given; that event will happen once in* n *trials, taking one time with another.*

For taking an infinite number of trials, that event in the whole will happen once in every n trials, becauſe all chances fall out with equal facility or difficulty. And conſequently the more trials the more it will approach to that ratio. And in an infinite number of repetitions, it approaches infinitely near a certainty, to fall out once in n trials. And therefore at a medium we ought to reckon, that this one chance will fall out in n trials.

Cor. 1. *If there be* n *equal chances in all; it is an even wager that a given chance will fall out in* $\frac{1}{2}n$ *trials. Or, there is an equal probability for its happening or failing in* $\frac{1}{2}n$ *trials, taking one time with another.*

For in a great number of repetitions of n trials, one chance will fall out for every repetition of n trials, or in every 2 repetitions of $\frac{1}{2}n$ trials. Therefore in theſe 2 repetitions one will happen and one will

will fail continually; and therefore it becomes equally probable, whether it will happen or fail in $\frac{1}{2}n$ trials.

Cor. 2. *Hence, to know how any event will fall out at a medium; it will be proper to conſider what the reſult will be, by an infinite number of trials.*

For innumerable trials take away all irregularities, and ſet all things in due proportion.

PROP. IX. PROB.

If a be the chances or number of caſes for happening, and b the number for the failing of an event. To find at how many trials, it is equally probable to happen or fail.

Let x be the number of trials. Then (Cor. 1. Prop. IV.) $1 - \frac{b^x}{\overline{a+b}^x}$ = probability of happening, which therefore is $= \frac{1}{2}$, whence $\overline{a+b}^x = 2b^x$. And $x \times \log. \overline{a+b} = \log. 2 + x \times \log. b$; therefore $x \times \log. \overline{a+b} - x \times \log. b = \log. 2 = .30103$, and $x = \frac{.30103}{\log. \overline{a+b} - \log. b} = \frac{.30103}{\log : \frac{a+b}{b}}$.

Cor. 1. *If* $a=b$, *then* $x=1$.

For $\log. \frac{a+b}{b} = \log. \frac{2a}{a} = \log. 2 = .30103$, and therefore $x = \frac{.30103}{.30103} = 1$.

Cor. 2. *If b is very great in reſpect of a, then* $x = \frac{7}{10} \times \frac{b}{a} = .7 \frac{b}{a}$, *nearly.*

For

For log. $\frac{a+b}{b}$ = log. 1 + $\frac{a}{b}$; where $\frac{a}{b}$ is a very ſmall quantity; and therefore by the nature of logarithms (Alg. Prob. 84. Cor. 1.) putting $v = \frac{a}{b}$, log. $1+v = .4343 \times v - \frac{vv}{2}$, &c. that is, log. $1 + \frac{a}{b} = .4343 \times \frac{a}{b}$, &c. therefore $x = \frac{.30103}{\text{log. } 1 + \frac{a}{b}} = \frac{.30103}{.4343 \times \frac{a}{b}} = \frac{.30103}{.4343} \times \frac{b}{a} = .693 \times \frac{b}{a} = .7 \times \frac{b}{a}$ nearly.

Cor. 3. *If b is very ſmall or* 0, *then x becomes very ſmall or nothing.*

Thus if $1 + \frac{a}{b}$ = 10, 100, 1000, 10,000, &c. then will $x = .3, \frac{.3}{2}, \frac{.3}{3}, \frac{.3}{4}$, &c. reſpectively.

PROP. X. PROB.

If there be a chances for happening, and b chances for failing of an event at 1 *trial; to find in how many trials it is equally probable or improbable for happening t times.*

If the trials be continued *ad infinitum*, the event will happen *a* times, and fail *b* times in every $a+b$ trials. And therefore one time with another it will happen once in every $\frac{a+b}{a}$ trials, and conſequently $\overline{t-1}$ times in $\overline{t-1} \times \frac{a+b}{a}$ trials. To this add the number of trials, in which there is an equal chance for happening once, which (by Cor. 2. Prop. IX.) is $\frac{.7b}{a}$, when *b* is large, and the number of trials

for

for happening n times will be $\frac{\cdot 7b}{a} + \overline{t-1} \times \frac{a+b}{a}$

$= \overline{t - \frac{3}{10}} \times \frac{b}{a} + t - 1.$

Therefore in $t - \frac{3}{10} \times \frac{b}{a} + t - 1$ trials, it is an equal chance whether the event will happen t times or not. But if a and b be nearly equal, the number of trials will be $\frac{1b}{a} + t - 1$.

This method is the more exact as b is greater.

Otherwiſe, more exactly.

Put $x =$ number of trials, then (by Prop. IV.) we ſhall have $\frac{a^x + xa^{x-1}b + x.\frac{x-1}{2}a^{x-2}bb + \&c.}{a+b^x} = \frac{1}{2}$;

the ſeries to be continued till the index of a be equal to t. Or elſe make $\frac{b^x + xb^{x-1}a + x.\frac{x-1}{2}b^{x-2}aa + \&c.}{a+b^x} = \frac{1}{2}$, continuing the ſeries till the index of a be $t-1$; and from either of them find the value of x. The former to be uſed when t is great, and the latter when t is ſmall. But when x is a large number, this will prove a very difficult calculation.

PROP. XI.

If A *and* B *play together, and* A *has* a *chances to win, and* B *has* b *chances. The probability of* A*'s winning, is to the probability of* B*'s winning, as* a *to* b.

This is the foundation of all gaming; for the more chances the greater the probability of winning; and the greater the probability the greater the ſucceſs, taking one time with another. Now the

the probability of A's winning is $\frac{a}{a+b}$, and that of B's winning, $\frac{b}{a+b}$; and theſe are as a to b.

Cor. 1. *Hence the odds in any wager laid upon a game, ſhould be as the chances; that is, the money laid on* A, *to the money on* B, *ſhould be as a to b, in a fair game. And the ſtakes ſhould alſo be in the ſame ratio.*

Cor. 2. *Hence alſo, one time with another,* A *will win a games, for* B*'s winning b games.*

PROP. XII.

To reſolve a Problem by the Laws of Chance.

1. The firſt thing to be done in any problem relative to chance, is to find out how many caſes there are in which the event will happen or fail; and likewiſe the number of all the caſes poſſible. So you will have the number of chances for and againſt you; this gives the probability of the event. But great care muſt be taken to allow no more nor fewer chances for an event, than really there are; for it is a frequent error to allow too many.

2. Oftentimes the probability of the failing of an event, or the number of caſes that fail, may be more eaſily found, than thoſe for the happening. Therefore find theſe firſt, and from thence thoſe for the happening will be eaſily found.

3. When ſeveral conditions are contained in the problem, each of which requires its proper degree of probability; then theſe ſeveral probabilities muſt be ſeparately found, and all added together for the total probability.

4. In

4. In many complicated problems, there will be ſo many caſes for the happening and failing of an event; that it will be a very tedious piece of work to count them up. In ſuch caſes, recourſe muſt be had to the doctrine of combinations and permutations, to obtain a complete ſolution.

5. Many problems require the help of Prop. V. and VI, eſpecially for two things A, B. In applying them, we muſt find the expectation, both when the event happens and fails, and take the ſum as the Prop. directs; and ſo proceed from the ſimple caſes to the harder.

SCHOLIUM.

Having laid down the foregoing propoſitions, containing the foundation of the Doctrine of Chances; I ſhall now give a few problems, to ſhew the uſe and application thereof in ſome eaſy caſes, as an introduction to that art; referring the reader to that excellent treatiſe of M. de Moivre's, for the determination of problems of greater difficulty.

PROB. I.

If I have ſix equal chances for 2, 3, 5, 8, 10, *and* 11 *ſhillings (and* 4 *chances for nothing) what is the value of my expectation.*

The probability of gaining each of theſe prizes, is $\frac{1}{10}$, therefore the ſum of my expectations is $\frac{1}{10} \times 2+3+5+8+10+11 = \frac{39}{10} = 3.9$ ſhillings the value required.

Or thus, $\frac{2+3+5+8+10+11}{10} = \frac{39}{10} = 3.9^{s}$, by Prop. V.

PROB.

PROB. II.

If I have 3 chances for 4 ſhillings, 6 chances for 8 ſhillings, and 5 chances for 5 ſhillings; what is the value of my expectation, theſe 14 being all the chances poſſible.

By Prop. VI. A, B, C, are reſpectively equal to 4, 8, 5, ſhillings; and a, b, c, equal to 3, 6, 5. Therefore my expectation is worth $\frac{3\times 4+6\times 8+5\times 5}{14}$ $= \frac{12+48+25}{14} = \frac{85}{14} = 6\frac{1}{14}$ ſhillings.

PROB. III.

To find the probability of caſting an ace in three throws with one die, or at one throw with three dice.

By Prop. VIII. the ace will be caſt in 6 throws, taking one time with another; and in 3 throws it will be an equal chance whether it be caſt or not. According to this the probability of caſting it in 3 throws is $\frac{1}{2}$, nearly.

Otherwiſe. By Prop. IV. put $a=1$, $b=5$, $n=3$, $t=1$; then $\frac{a^3+3a^2b+3abb}{a+b^3}$ = probability; that is, $\frac{1+15+75}{216} = \frac{91}{216}$.

Or, $\frac{b^3}{a+b^3} = \frac{125}{216}$ = probability of failing; and therefore $1-\frac{125}{216} = \frac{91}{216}$ is the probability of happening, as before.

PROB. IV.

To find the probability of caſting an ace in 6 throws with one die.

Though this event will happen in 6 trials, taking one time with another (by Prop. VIII.); yet in barely 6 trials it cannot be depended on. For in an infinite number of repetitions it continually approximates to a certainty of throwing an ace in 6 throws; yet in one repetition only, it has but a certain degree of probability. To find which,

Put $a=1$, $b=5$, $n=6$, $t=1$. Then (Prop. IV.) $\frac{b^6}{n^6}$ = probability of failing $= \frac{5^6}{6^6} = \frac{15625}{46656}$, and the probability of happening $= 1 - \frac{15625}{46656} = \frac{31031}{46656}$ $= \frac{2}{3}$ nearly.

Otherwiſe.

Let 1ll be the prize. At the 1ſt throw there is 1 chance for me and 5 againſt me, (and the like every throw). Therefore (by Prop. VI.) I have 1 chance for 1ll, and 5 for 0; therefore my expectation $= \frac{1\times 1+5\times 0}{6} = \frac{1}{6}$, for 1 throw.

For 2 throws. If I hit the firſt time I gain 1, if I miſs it, I have 1 throw remaining $= \frac{1}{6}$. Therefore I have 1 chance for 1, and 5 for $\frac{1}{6}$, and my expectation $= \frac{1+5\times\frac{1}{6}}{6} = \frac{11}{36}$.

For

For 3 throws; my expectation $\frac{1+5\times\frac{11}{36}}{6} = \frac{91}{216}$, as in Prob. III.

Proceeding thus to 6 throws, we get $\frac{1+5\times\frac{4651}{7776}}{6}$ $= \frac{31031}{46656}$, as before.

PROB. V.

In how many throws may one undertake for an even wager to caſt an ace with one die?

Let x = number of throws, $a = 1$ = chance to hit, $b = 5$ = chances to miſs. Then (Cor. 1. Pr. IV.) $\frac{b^x}{\overline{a+b}^x}$ = probability of failing, which (in an even wager) is equal to the probability of happening; therefore $\frac{b^x}{\overline{a+b}^x} = \frac{1}{2}$, and $2b^x = \overline{a+b}^x$.

Whence log. 2 + log. b^x = log. $\overline{a+b}^x$, or log. 2 + x × log. $b = x \times$ log. $\overline{a+b}$; and $x \times$ log. $\overline{a+b} - x \times$ log. b = log. 2. Therefore $x = \frac{\text{log. } 2}{\text{log. } \overline{a+b} - \text{log. } b} = \frac{\text{log. } 2.}{\text{log. } 6 - \text{log. } 5} = \frac{.30103}{.07918}$ = 3.82, that is, between 3 and 4 throws; or nearly at 4 throws, one may undertake to throw an ace.

Otherwiſe, proceed (by Prop. VI.) as in the laſt example, till the expectation be = $\frac{1}{2}$.

PROB. VI.

In how many throws with one die, may one undertake to throw an ace 4 times?

Let x = number of throws, $a=1$, $b=5$, $t=4$, then proceeding as before, we ſhall have (by Prop. IV.)

$$\frac{a^x+xa^{x-1}b+x.\frac{x-1}{2}a^{x-2}bb \text{ &c.}}{\overline{a+b^x}} = \frac{1}{2}, \text{ or } a^x+xa^{x-1}b$$

$$+x.\frac{x-1}{2}a^{x-2}bb \text{ &c. till } a_4 = \frac{1}{2}\times\overline{a+b^x} = \frac{\overline{a+b^x}}{2}.$$

To find x, the eaſieſt way is to take it by gueſs, and add 1 every trial, which will increaſe a term of the ſeries every time; proceed till you find the two ſides of the equation nearly equal, and then you have the value of x nearly. But to ſhorten the work, proceed thus. Since an ace may be thrown every 6 throws, taking one time with another for a great many repetitions; and by Prob. V. at one turn only, (or a few turns) it will happen once in 3.82 throws; therefore the ace to be thrown 4 times, requires at leaſt 4 times 3.82 throws, or 15.28 throws; but to be thrown at 6 throws always, requires 4 times 6, or 24 throws at moſt. Therefore x will probably be between 15 and 24. And therefore I ſhall ſuppoſe it, at a mean, to be 19. Then we have $a^{19}+19a^{18}b+19.\frac{18}{2}a^{17}bb$ &c. $=\frac{\overline{a+b^{19}}}{2}$. But ſince here will be a great number of terms before we come at the index a^4; therefore it will be ſhorter to take the other end of the ſeries, for the failing; ſince in an equal wager, they are of equal value; and then we have $b^x+xb^{x-1}a$ $+x.\frac{x-1}{2}b^{x-2}a^2+x.\frac{x-1}{2}.\frac{x-2}{3}b^{x-3}a^3 = \frac{\overline{a+b^x}}{2}$: and thus we have but 4 terms to add up. In the preſent ſuppoſition $5^{19}+19\times5^{18}+171\times5^{17}+969\times5^{16}$ $=$

$= \frac{6^{19}}{2}$, where a is $=1$. Theſe put into numbers by help of logarithms, will give 517, which ſhould be 305, neglecting ſuperfluous numbers or places.

Again, taking $x = 20$, we get the ſeries 207, which ſhould be 183, therefore x is ſtill too little; therefore,

Take $x = 21$, then the ſeries is 115, which is nearly $=109$, or $= \frac{6^{21}}{2}$. Again,

Take $x=22$, and the ſeries is $= 642$, which is ſomething more than 656, the 22d power of 6 divided by 2; therefore here x is too great.

Whence we find x between 21 and 22, being very little more than 21, the number of throws required.

Cor. 1. *Hence in this example, the number of throws required is nearly ſo many times* 6, *abating one,* $+3.8 = 18+3.8 = 21.8$. *And probably is nearly the ſame in other like caſes.*

As if it were required to find how many throws are required to throw an ace 6 times; the anſwer is $5 \times 6 + 3.8 = 33.8$, or near it.

Cor. 2. *And hence may be collected, at how many throws, the ſame may be done with ſeveral dice.*

As if it be required to find in how many throws one may undertake to throw 4 aces with 6 dice. Having found the number of throws with one die, 21; divide it by the number of dice; ſo $\frac{21}{6} = 3\frac{1}{2}$, the throws required with 6 dice.

SCHOLIUM.

Many queſtions in chance of this ſort, require ſuch an infinite deal of labour in calculating, that

few people can have the patience to go through it. And therefore it is better in ſuch caſes to be content with ſuch approximations as come near the matter, ſince chance is but chance; or, in other words, ſince chance affords nothing certain; hardly any thing ever falling out as it is calculated.

LEMMA.

To find the number of caſes to throw a given number, with any number of dice.

Let the given number of points $= p + 1$,
$n =$ number of dice,

Then the number of caſes is

$$+ \frac{p}{1} \times \frac{p-1}{2} \times \frac{p-2}{3}, \text{ \&c. (to } n - 1 \text{ terms)}$$
$$- \frac{p-6}{1} \times \frac{p-7}{2} \times \frac{p-8}{3}, \text{ \&c. } \times : \frac{n}{1}$$
$$+ \frac{p-12}{1} \times \frac{p-13}{2} \times \frac{p-14}{3} \times : \frac{n}{1} \times \frac{n-1}{2}$$
$$- \frac{p-18}{1} \times \frac{p-19}{2} \times \frac{p-20}{3} \times : \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$$
$$+ \text{ \&c.}$$

Which rows of ſeries muſt be continued downwards, till ſome of the factors in each become o or negative; taking $n - 1$ factors in each.

This is demonſtrated by Mr. D' Moivre, but I have no room for it in this place.

EXAM.

EXAM.

How many chances are there for throwing 16 with 4 dice.

Here $p=15$, $n=4$, whence

$$+\frac{15}{1}\times\frac{14}{2}\times\frac{13}{3} = 455$$
$$-\frac{9}{1}\times\frac{8}{2}\times\frac{7}{3}\times\frac{4}{1} = -336$$
$$+\frac{3}{1}\times\frac{2}{2}\times\frac{1}{3}\times\frac{4}{1}\times\frac{3}{2} = +6$$

Sum 125

And the whole number of chances is 6^4 or 1296.

PROB. VII.

What is the probability of casting 15 *points with* 6 *dice?*

Here $p=14$, $n=6$, whence

$$+\frac{14}{1}\times\frac{13}{2}\times\frac{12}{3}\times\frac{11}{4}\times\frac{10}{5} = +2002$$
$$-\frac{8}{1}\times\frac{7}{2}\times\frac{6}{3}\times\frac{5}{4}\times\frac{4}{5}\times 6 = -336$$

Numb. points 1666

But $6^6 = 46656$; therefore the probability is $\frac{1666}{46656}$.

PROB. VIII.

A *undertakes to throw* 10 *with* 3 *dice, before* B *throws* 12 *with* 4 *dice. What is the odds of the game?*

First for the probability of caſting 10 with 3 dice; here $p = 9$, $n = 3$, then

$$+\frac{9}{1}\times\frac{8}{2} = +36$$

$$-\frac{3}{1}\times\frac{2}{2}\times\frac{3}{1} = -9$$

Number of caſes 27

But the whole number of caſes is 6^3 or 216, therefore the probability at one throw is $\frac{27}{216}$ for A.

Secondly, for the probability of caſting 12 with 4 dice. Here $p = 11$, $n = 4$, therefore

$$+\frac{11}{1}\times\frac{10}{2}\times\frac{9}{3} = +165$$

$$-\frac{5}{1}\times\frac{4}{2}\times\frac{3}{3}\times\frac{4}{1} = -40$$

Number of caſes 125

But here the whole number of caſes is 6^4, or 1296, therefore B's probability at one throw is $\frac{125}{1296}$. Whence A's chance is to B's as $\frac{27}{216}$ to $\frac{125}{1296}$, or as $\frac{27}{6^3}$ to $\frac{125}{6^4}$ or $\frac{162}{6^4}$ to $\frac{125}{6^4}$, that is, as 162 to 125; ſo that A has the better of the wager.

PROB. IX.

Any number (n) of dice being given; to find the probability of caſting 2 faces of a ſort, at one throw.

In two dice, to avoid having 2 faces of a ſort, the 6 faces of one die can only be combined with 5 faces

5 faces of the other, and this number of combinations will be $6 \times 5 = 30$.

In three dice, the 30 throws of 2 dice, which have no two of a ſort, can only be combined with 4 of the third die; for being combined with 5, there would be ſeveral times two of a ſort. And the number of theſe combinations will be $6 \times 5 \times 4 = 120$.

Again, in 4 dice, to avoid 2 of a ſort, the 120 of 3 dice can only be combined with 3 of the 4th die; and theſe combinations are $6 \times 5 \times 4 \times 3 = 360$, and ſo on.

Therefore there will be had the probability of failing; which will be, for 2 dice, $\frac{6 \times 5}{6 \times 6}$; for three dice $\frac{6 \times 5 \times 4}{6 \times 6 \times 6}$; for four dice $\frac{6 \times 5 \times 4 \times 3}{6 \times 6 \times 6 \times 6}$, &c. and univerſally for n dice, $\frac{6 \times 5 \times 4 \text{ (to } n \text{ terms)}}{6 \times 6 \times 6 \text{ (to } n \text{ terms)}}$; becauſe $6 \times 6 \times 6$ to n terms is the whole number of chances. Whence we have the probability of happening $= 1 - \frac{6.5.4}{6.6.6}$ (to n terms.)

EXAMPLE I.

What is the probability of caſting 2 faces of a ſort with 4 dice.

Here $\frac{6.5.4.3}{6.6.6.6} = \frac{360}{1296}$ the probability of failing, and $\frac{1296 - 360}{1296} = \frac{936}{1296}$ the probability of happening.

EXAM. II.

What is the probability of throwing 2 faces of a ſort with 6 dice.

Here

Here $\frac{6.5.4.3.2.1}{6.6.6.6.6.6} = \frac{120}{7776}$ the probability of failing, and $\frac{7656}{7776}$ the probability of happening.

EXAM. III.

To find the probability of throwing two faces of a ſort with 7 dice.

Here $\frac{6.5.4.3.2.1.0}{6.6.6.6.6.6.6} = \frac{0}{46656}$ the probability of failing, and $\frac{46656}{46656} = 1$ the probability of happening, which is a certainty.

Therefore in 7 dice or more, there muſt neceſſarily be two of a ſort.

PROB. X.

In how many throws may one undertake to throw all the faces of a die.

It is impoſſible to perform this in leſs than ſix throws. And taking one time with another, the old faces are as likely to come up as new ones, which will require twice the number at leaſt. Therefore to throw all the faces, will require at leaſt 12 caſts.

Otherwiſe.

At the firſt throw ſome of the faces muſt neceſſarily be caſt. At the 2d throw the probability of caſting a different face is $\frac{5}{6}$. At the 3d throw, the probability of caſting one of the remaining faces is $\frac{4}{6}$. And at the 4th, 5th, and 6th throws, the

the probability of ſtill caſting one of the remaining faces will be $\frac{3}{6}$, $\frac{2}{6}$, and $\frac{1}{6}$, reſpectively. Then to find the number of throws requiſite for each.

The probability $\frac{5}{6}$ requires—$1\frac{1}{5}$ throws.

The probability $\frac{4}{6}$ requires—$1\frac{1}{2}$ throws.

The probability $\frac{3}{6}$ requires—2 throws.

The probability $\frac{2}{6}$ requires—3 throws.

The probability $\frac{1}{6}$ requires—6 throws.

in all	13.7
for the firſt add	1.0
	14.7

Therefore in 14 throws all the faces will probably be caſt.

Mr. D' Moivre by a difficult calculation finds 13 throws for caſting all the faces, which doubtleſs is more exact than mine, which are only ſo many approaches towards truth.

PROB. XI.

A perſon undertakes with 4 *dice to throw* 12, 13, 14, *or* 15 *every throw. What is the odds?*

By the Lemma (Prob. VII.) we ſhall find,

The chances for 12	—	125
The chances for 13	—	140
The chances for 14	—	146
The chances for 15	—	140
All the chances for him		551

All

All the chances on 4 dice $= 4^6 = 1296$.
And $1296 - 551 = 745$ the chances againſt him.
Then the odds againſt him is as 745 to 551.

PROB. XII.

To find the probability of caſting m aces only, at n throws with one die, or at 1 throw with n dice.

Here n muſt be equal to, or greater than m. Let C = combinations of all the aces, taken by ones, or by twos or threes, &c. Then,

1. Suppoſe only one ace to be caſt, then $m = 1$, therefore ſetting aſide one die, the remaining dice will be $n - 1$; and to avoid aces with theſe, only 5 points of each muſt be taken. And the number of combinations of all theſe will be 5^{n-1}, which combined with all the aces in each ſingle die, will be $C \times 5^{n-1}$, where one ace is caſt.

2. Suppoſe $m = 2$, then ſetting aſide 2 dice, the number of combinations without any ace in $n - 2$ dice, will be 5^{n-2}; and theſe, combined with every 2 aces in the remaining dice, will be $C \times 5^{n-2}$.

3. If $m = 3$, the combinations without any ace will be 5^{n-3}, and theſe combined with every three, will be $C \times 5^{n-3}$.

And in general, the combinations, without any ace, in $n - m$ dice will be 5^{n-m}; and theſe combined with all the m's out of n aces, will be $C \times 5^{m-n}$ for the number of caſes that m aces can be caſt, excluding all aces whatever, beſides theſe.

Now by the doctrine of combinations, when $m = 1$, $C = n$, and when $m = 2$, $C = \frac{n}{1} \cdot \frac{n-1}{2}$. And when $m = 3$, $C = \frac{n.n-1.n-2}{1 \cdot 2 \cdot 3}$, and in general, for

m aces,

m aces, $C = \frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}}{1 \cdot 2 \cdot 3 \cdot 4}$ (to m terms). And since 6^n is the number of all the chances on n dice; therefore $\frac{n}{1} \cdot \frac{\overline{n-1}}{2} \cdot \frac{\overline{n-2}}{3}$ (to m terms) $\times 5^{m-n}$, is the number of all the caſes that m aces only can be caſt, and $\frac{n}{1} \cdot \frac{\overline{n-1}}{2} \cdot \frac{\overline{n-2}}{3}$ (to m terms) $\times \frac{5^{n-m}}{6^n}$ = probability of caſting them.

EXAMPLE.

What is the probability of caſting 3 aces only, with 10 dice; or with 10 throws of 1 die.

Here $m=3$, $n=10$, and the number of chances is $\frac{10}{1} \cdot \frac{9}{2} \cdot \frac{8}{3} \times 5^7 = 120 \times 78125 = 9375000$. And $6^{10} = 60466176$; therefore the probability is $= \frac{9375000}{60466176} = \frac{2}{13}$ nearly.

PROB. XIII.

What is the probability of throwing both an ace and a ſix, with three dice.

By Prop. IV. the probability of caſting an ace once in three throws of 1 die, or in one throw of 3 dice, is $1 - \frac{b^3}{a+b^3} = \frac{\overline{a+b^3} - b^3}{a+b^3} = \frac{6^3-5^3}{6^3}$, where $a=1$, $b=5$. And the number of chances $= 6^3-5^3$.

Set aſide the ſix, then for the ſame reaſon the number of chances for the ace is $5^3 - 4^3$.

Reſtore the ſix, and the chances for the ace to come up without the ſix, will be the ſame as if the ſix was wanting.

Now

Now from the chances for the ace to come up, with or without the ſix ($6^3 - 5^3$), ſubtract the chances for the ace to come up without the ſix ($5^3 - 4^3$); there will remain the chances for both the ace and ſix to come up ($6^3 - 2 \times 5^3 + 4^3$. Conſequently $\frac{6^3 - 2 \times 5^3 + 4^3}{6^3} = \frac{30}{216} = \frac{5}{36}$ is the probability required.

SCHOL.

By the ſame method $\frac{6^4 - 3 \times 5^4 + 3 \times 4^4 - 3^4}{6^4}$ will be the probability of caſting 3 given faces with 4 dice.

PROB. XIV.

A, B, C, *play with two dice, the higheſt number take all; and* A *has caſt* 8. *What is the value of his expectation for the prize* S?

The mean number on two dice is 7, which muſt be reckoned for B and C, without caſting. Therefore S is to be divided in proportion to the numbers 8, 7, 7, for A, B, C. Whence A's ſhare is $\frac{8}{22}$ S $= \frac{4}{11}$ S.

PROB. XV. (L. Diary, 1741.)

A perſon with ſix dice undertakes to bring up four faces of a ſort at one throw, in ſeven trials. What is the odds againſt him?

The number of combinations of 4 aces out of 6 is $\frac{6.5.4.3}{1.2.3.4} = 15$. And in any one caſe of theſe

15,

15, any two left out are capable of 25 variations, where no ace is found; therefore 25×15, or 375, is the number of caſes where only 4 aces can be caſt. But becauſe 5 or 6 aces may be caſt, the number of chances for theſe muſt alſo be added.

Now the number of combinations of 5 aces out of 6, is 6; and ſince any one left out is capable of 5 variations where no ace is found; therefore 6×5 or 30 is the number of caſes wherein only 5 aces can be caſt with 6 dice. Laſtly, there is but 1 caſe wherein 6 aces can be caſt with 6 dice; therefore the ſum of all the caſes wherein 4 aces can be caſt with 6 dice is $375 + 30 + 1 = 406$.

Now ſince there is the ſame variety for duces, trays, &c. therefore $406 \times 6 = 2436$ is the whole number of caſes wherein 4 points of any ſort can be caſt. Put $s = 6^6 = 46656$, and $46656 - 2436 = 44220 = q$; then (by Cor. 1. Prop. IV.) the odds againſt the thrower will be as $\overline{\frac{q}{s}}^7$ to $1 - \overline{\frac{q}{s}}^7$, that is, as .68705 to .31295, or as 2.1954 to 1.

PROB. XVI.

A undertakes with 5 dice, to throw 3 aces, a three, and a four at a throw; before B *throws 2 duces, a five, and 2 ſixes. To find their probabilities of winning.*

All the chances on 5 dice amount to $6^5 = 7776$. Let a, b, c, d, e, f, ſtand for the ace, duce, 3, 4, 5, and 6. Then A's chance is a^3cd; and B's chance is b^2ef^2, where the indices only ſhew the repetitions of each letter. The number of permutations of 5 things a^3cd is $\frac{1.2.3.4.5}{1.2.3.1.1} = 20$ (by Prop. IV. Permutations.) And the number of permutations of the 5 things

5 things b^2ef^2 is $\frac{1.2.3.4.5}{1.2.1.1.2} = 30$. Therefore A's chance of winning, is to B's, as $\frac{20}{7776}$ to $\frac{30}{7776}$, or as 2 to 3.

PROB. XVII.

To find how many ways, three aces, two duces, a tray, and a four, can be caſt with ſeven dice.

Let a, b, c, d, e, f, ſignify the ace, duce, tray, four, five and ſix, of any die. And ſince one face of every die is to appear, ſet down ſeven rows of them. Then a^3b^2cd will expreſs the numbers.

a	b	c	d	e	f
a	b	c	d	e	f
a	b	c	d	e	f
a	b	c	d	e	f
a	b	c	d	e	f
a	b	c	d	e	f
a	b	c	d	e	f

Now the three a's will fall in ſome of the ſeven rows, the two b's in ſome of the remaining four, the c in one of the remaining two, and the d in the remaining one row, as they are ſet off in the figure.

The number of combinations of 3 a's out of 7, is $\frac{7.6.5}{1.2.3} = 35$, and ſo often may 3 a's be combined with the remaining 4 b's.

The number of combinations of 2 b's out of 4 is $\frac{4.3}{1.2} = 6$; therefore the number of combinations of the a's and b's is 35×6. And ſo often may the a's and b's be combined with either of the two c's.

The number of combinations of 1 c out of 2, is $\frac{2}{1} = 2$, therefore the number of combinations of the

the a's, b's, and c's, is $= 35 \times 6 \times 2$. And lastly, the combination of $1\,d$ is but 1; therefore the number of all the combinations is $= 35 \times 6 \times 2 \times 1$, or $\frac{7.6.5}{1.2.3} \times \frac{4.3}{1.2} \times \frac{2}{1} \times \frac{1}{1} = \frac{1.2.3.4.5.6.7}{1.2.3 \times 1.2 \times 1 \times 1} =$ 420, all the ways these can be cast, or the number of chances. But the number of all the chances on 7 dice is 6^7, or 279936.

Cor. *Hence the probability of casting 3 aces, 2 duces, 1 three, and 1 four, with 7 dice, at a throw is* $\frac{420}{279936} = \frac{2}{1333}$ *nearly.*

In general.

Let $a^p b^q c^r d^s$ &c. be proposed, where p, q, r, &c. denote the number of aces, duces, &c. or of any other faces that are named. Let $p+q+r+s$ &c. $= n =$ the number of dice.

Then observing the process that went before, we shall have $\frac{1.2.3.4 \text{ (to } n \text{ terms)}}{1.2.3(p \text{ terms}) \times 1.2.3(q) \times 1.2.3(r) \text{ \&c.}}$ = number of all the cases for casting p aces, q duces, &c.

PROB. XVIII.

To find how many ways 3 faces of one sort, 2 of another sort, 1 of a third sort, and 1 of a fourth, can be cast with 7 dice.

Let $a^3 b^2 c^1 d^1$ denote the quantities, where a, b, c, d, stand for any four different faces; the indices 3, 2, 1, 1, the number of each.

Set down the 7 rows as in the last problem. Then it is plain, the combination of 3 a's out of

$7 = \frac{7.6.5}{1.2.3}$; and that of 2 b's out of $4 = \frac{4.3}{2}$; that of 1 c out of $2 = 2$, therefore we have $\frac{7.6.5}{1.2.3} a^3 \times \frac{4.3}{1.2} < \frac{2}{1} c \times d$. But ſince a, b, c, d, may ſtand for any ſort, there will be 6 variations for a^3 (viz. a^3, b^3, c^3, d^3, e^3, f^3), and 5 variations for b^2; 4 variations for c, and 3 for d, and the combinations of c and $d = \frac{4.3}{2}$. Then the whole number of combinations will be $\frac{7.6.5}{1.2.3} \times 6 \times \frac{4.3}{1.2} \times 5 \times \frac{2}{1} \times \frac{4.3}{2} = \frac{7.6.5.4.3.2.1}{1.2.3 \times 1.2 \times 1} \times \frac{6.5.4.3}{1.2} = 420 \times 180 = 75600$.

Hence in general, let G = number of caſes, when all the faces are named (as in Prob. XVII); l = number of quantities mentioned, and $H = \frac{6.5.4 \text{ } (l \text{ terms})}{R \times S \times T \text{ \&c.}}$. Then GH = number of caſes ſought, where R = 1.2 when any index is twice repeated, or 1.2.3, when thrice repeated, &c. and the like for S, T, &c.

Exam.

Suppoſe a^3b^2cd as before, then G = 420, and $l = 4$, R = 1, S = 1, T = 1.2. Therefore GH $= 420 \times \frac{6.5.4.3}{1.1.1.2} = 420 \times 180 = 75600$.

PROB. XIX.

If A *and* B *be playing together, and* A *has got* 1 *of the game; what advantage has he of the game?*

If

If 2 be up, then A wants 1, and B wants 2; and therefore A's chance againſt B, is as 2 to 1.

If they be up in 1000, 10,000, &c. then A has no advantage over B, and their chances are equal, or as 1 to 1.

Therefore A's chance is always between 1 and 2, to B's 1. And the fewer games up, the nearer it approaches to 2 to 1, againſt B.

Otherwiſe, more exactly.

Let n be the number for being up, and A has got 1, put $\frac{1.3.5.7}{2.4.6.8}$ (to $n-1$ terms) $= x$; then A's advantage to B's, is as $1+x$ to $1-x$; and if they leave off play, the money muſt be divided in that proportion.

Exam. Suppoſe 4 be up, then $x = \frac{1.3.5}{2.4.6} = \frac{5}{16}$, then A's ſhare is $\frac{21}{16}$, and B's $\frac{11}{16}$, or rather $\frac{21}{32}$ and $\frac{11}{32}$.

PROB. XX.

A and B *playing together,* A *wants* 2 *of being up, and* B *wants* 3; *what is the advantage of either?*

A's advantage or chance to B's chance, is as 3 to 2 nearly.

Or thus, more exactly.

It is evident the play will be ended in 4 games. Take all the uncice of the 4th power of a binomial, $1 + 4 + 6 + 4 + 1$; take 2 of the firſt terms, $1 + 4 = 5$, for A; and 3 of the laſt terms for B $= 1 + 4 + 6 = 11$; and A's chance is to B's reciprocally as 11 to 5.

Or thus, more generally.

1. Let 1*l.* be the ſtake; and ſuppoſe each perſon wants 1 of being up, then the expectation of each (by Prop. V.) is $\frac{1}{2} \times 1l. = \frac{1}{2}$.

2. If A wants 1 of being up, and B 2. If A gets 1, he gets the ſtake 1; if B gets 1, then each wants 1 of being up, and they are in the ſame condition as in Art. 1. and A's expectation is $\frac{1}{2}$. Therefore (before the game be played) A has an equal chance to get the whole or the half, and therefore A's expectation $= \frac{1 + \frac{1}{2}}{2} = \frac{3}{4}$. And B's $= \frac{1}{4}$.

3. Let A want 1, and B 3. If A gets one he is up. If B gets 1 he will want 2 of being up; and they are in the ſame ſtate as Art. 2. therefore A's expectation $= \frac{3}{4}$. Therefore (before the play) A has an equal chance for 1 or $\frac{3}{4}$, and his expectation $= \frac{1 + \frac{3}{4}}{2} = \frac{7}{8}$.

4. Let A want 1 and B 4; if A gets 1 he is up; if B gets 1, he wants 3, which is the caſe of Art. 3. and A's expectation is then $= \frac{7}{8}$, whence A has an equal chance for 1 or $\frac{7}{8}$; and his expectation $= \frac{1 + \frac{7}{8}}{2} = \frac{15}{16}$.

5. If A wants 1 and B 5; if A gets 1 he is up; if B gets 1, he wants 4, in which caſe (Art. 4.) A's expectation is $\frac{15}{16}$. Whence (before the play) he has an equal chance for 1 or $\frac{15}{16}$. Therefore his expectation $= \frac{1 + \frac{15}{16}}{2} = \frac{31}{32}$, and ſo forward.

6. Hence in general, if A wants 1, and B n of being up, then A's expectation $= \frac{2^n - 1}{2^n}$.

Again,

7. Suppoſe each wants 2 of being up; then the expectation of each is $= \frac{1}{2}$.

8. Sup-

8. Suppose A wants 2, and B 3; if A gets 1, then he will want 1 and B 3, then (Art. 3.) in that case, A's chance is worth $\frac{7}{8}$. But if B gets 1, they will both want 2, and then A's expectation $= \frac{1}{2}$. So that (before the play) A has an equal chance for $\frac{7}{8}$ or $\frac{1}{2}$. Whence A's expectation is $= \frac{\frac{1}{2} + \frac{7}{8}}{2} = \frac{11}{16}$.

9. If A wants 2 and B 4; if A gets the game, he wants 1 and B 4, as in Art. 4, where his exp. $= \frac{15}{16}$. But if B gets 1, then A wants 2 and B 3, as in Art. 8, where his exp. $= \frac{11}{16}$. Whence he has an equal chance for $\frac{15}{16}$ or $\frac{11}{16}$. And his expectation $= \frac{\frac{15}{16} + \frac{11}{16}}{2} = \frac{26}{32} = \frac{13}{16}$.

10. If A wants 2 and B 5; if A gets 1, he will want 1 and B 5; but if B gets 1, A will want 2 and B 4. Therefore (Art. 5, 9,) A will have an equal chance for $\frac{31}{32}$ or $\frac{26}{32}$, and his expectation $= \frac{\frac{31}{32} + \frac{26}{32}}{2} = \frac{57}{64}$.

11. If A wants 2 and B 6. If A gets 1, he will want 1 and B 6. But if B gets 1, A will want 2 and B 5. Whence (Art. 6, 10.) A has an equal chance for $\frac{63}{64}$ or $\frac{57}{64}$; and A's expectation $= \frac{120}{128}$. Again,

12. If A wants 3 and B 4. If A gets 1, then he wants 2 and B 4. But if B gets 1, then A wants 3 and B 3 (whose chances are equal $= \frac{1}{2}$). Therefore (Art. 11.) A has an equal chance for $\frac{13}{16}$ or $\frac{1}{2}$, and his expectation $= \frac{\frac{1}{2} + \frac{13}{16}}{2} = \frac{21}{32}$. And thus you may proceed as far as you will.

PROB. XXI.

A and B *play together*; A *wants* 2 *and* B 3 *of being up*; *and* A's *dexterity or ſkill is to* B's, *as* 4 *to* 3; *what is the odds of the game?*

Since the play will be ended in 4 games; involve $a+b$ to the 4th power, $a^4 + 4a^3b + 6a^2bb + 4ab^3 + b^4$, where $a = 4$, $b = 3$. Then for A take all the terms where a is of 2 or more dimenſions $a^4 + 4a^3b + 6a^2b^2 = 1888$; and for B, thoſe where b is of 3 or more dimenſions, $4ab^3 + b^4 = 513$; then A's chance is to B's, as 1888 to 513.

Otherwiſe.

Divide the ſkill, by the defect from being up; for the chances of each. Thus A's chance is to B's, as $\frac{4}{1}$ to $\frac{3}{1}$, or as 2 to 1 nearly; but not ſo exact as the former.

PROB. XXII.

Three gameſters A, B, C, *play together, whoſe proportion of ſkill is as a, b, c; they play till* A *wants* 1, B 2, C 3, *of being up. What are their ſeveral chances?*

Here the play will be concluded in 4 games; therefore raiſe the binomial $a + b + c$, to the 4th power, which will be $a^4 + 4a^3b + 6aabb + 12abbc + 4b^3c + 4bc^3 + 4ab^3 + b^4 + 4a^3c + 12a^2bc + 6a^2c^2 + 12abcc + 6bbcc + 4ac^3 + c^4$.

Take all the terms for A, where a is of 1 or more dimenſions, (where he wants 1) and where b and c are of fewer dimenſions than 2 and 3 (the numbers they want). So will A's chance be $a^4 + 4a^3b + 12aabc + 4a^3c + 12abcc + 6aacc$.

And B's $b^4 + 4b^3c + 6bbcc$.

And C's $4bc^3 + c^4$.

The

The reſt of the terms muſt be divided among them, according as they favour one or another of the players.

Otherwiſe.

Take the proportion of their ſkill, which divide reſpectively by what they want of being up. Thus the chances of A, B, C, will be $\frac{a}{1}, \frac{b}{2}, \frac{c}{3}$, nearly.

PROB. XXIII.

A and B *playing together,* A *has a chances for the game, and* B *has b chances; alſo* A *ſtakes p, and* B *ſtakes q; to find the advantage or diſadvantage of the players.*

Since A has a chances, and B has b, in any game; therefore in playing $a+b$ games, A will win a games, and B will win b. Therefore,

A gets $a \times \overline{p+q}$, and wins $a \times \overline{p+q} - \overline{a+b} \times p = aq - bp$.

B gets $b \times \overline{p+q}$, and wins $b \times \overline{p+q} - \overline{a+b} \times q = bp - aq$.

Therefore the advantage of A is $aq - bp$; and that of B, $bp - aq$. Therefore if $a : b :: p : q$, then $aq = bp$, and $aq - bp$, and alſo $bp - aq = 0$, and neither of them has the advantage. But if q is greater, then A has the advantage; but if p is greater, then B has the advantage.

PROB. XXIV.

A and B *playing together with n bowls a-piece; to find the probability of* A*'s getting all his bowls at at end.*

Let a, b, c, d, e, f, &c. be all the bowls that both have, being equal to $2n$ in number; a, b, c, being

A's, and *d*, *e*, *f*, B's. The number of permutations that they are all capable of is 1.2.3.4 (to $2n$ terms,) being the number of all the chances. The number of all the permutations, that the bowls *d*, *e*, *f*, (furtheſt from the jack) are capable of is 1.2.3 (to n terms), when at the ſame time *a*, *b*, *c*, are neareſt the jack. Theſe are the number of chances for any one poſition of the bowls *a*, *b*, *c*; when *d*, *e*, *f*, are furtheſt from the jack. But ſince *a*, *b*, *c*, are likewiſe capable of the ſame number of permutations 1.2.3 (to n terms), therefore 1.2.3 (to n terms) × 1.2.3 (to n terms) is the number of all the chances for *a*, *b*, *c*, being neareſt, and *d*, *e*, *f*, furtheſt from the jack. Therefore $\frac{1.2.3\ (n \text{ terms}) \times 1.2.3\ (n \text{ terms})}{1.2.3.4\ (\text{to } 2n \text{ terms})}$ = the probability required; that is, $\frac{1.2.3\ (\text{to } n \text{ terms})}{n+1.n+2.n+3\ (\text{to } n \text{ terms})}$ = probability. Or thus, $\frac{n}{2n} \times \frac{n-1}{2n-1} \times \frac{n-2}{2n-2}$ (to n terms) = probability of all A's bowls being neareſt the jack.

PROB. XXV.

A and B *playing together with n bowls a-piece, to find the probability of* A*'s getting m bowls at an end.*

The number of permutations of $2n - m$ bowls, (the number furtheſt from the jack) is 1.2.3 ... to $2n-m$. Here are n bowls of B, and $n - m$ bowls of A. But the number of combinations of $n - m$ things out of n, is $\frac{n.n-1}{1.2}$ (to $n - m$ terms); and ſo many different ways, will theſe permutations be varied, all which will become $\frac{n.n-1}{1.2}$ (to $n - m$ terms)

terms) $\times$ 1.2.3 ... to $2n - m$. Likewiſe the permutations of A's m bowls, will be 1.2.3 ... to m; and the number of permutations of $2n$ things being 1.2.3 ... to $2n$, therefore the probability required is

$$\frac{1.2.3 \text{ (to } m \text{ terms)} \times 1.2.3 \ldots \text{(to } 2n-m) \times \frac{n.n-1}{1.2} \ (n-m \text{ terms})}{1.2.3 \ldots \text{ to } 2n} =$$

$$\frac{\frac{1.2.3\ (m)}{1.2.3\ (n-m \text{ terms})} \times \frac{1.2.3 \ldots \text{ to } 2n}{2n.2n-1\ (m \text{ terms})} \times n.n-1\ (n-m \text{ terms})}{1.2.3 \ldots \text{ to } 2n} =$$

$$\frac{1.2.3\ (m)}{1.2.3\ (n-m \text{ terms})} \times \frac{n.n-1\ (n-m \text{ terms})}{2n.2n-1\ (m \text{ terms})} =$$

$$\frac{1.2.3\ (m)}{1.2.3\ (n-m \text{ terms})} \times \frac{\frac{n.n-1 \ldots \text{ to } 1}{1.2.3\ (m)}}{2n.2n-1\ (m \text{ terms})} =$$

$$\frac{n.n-1 \ldots \text{ to } 1}{1.2.3\ (n-m \text{ terms}) \times 2n.2n-1\ (m \text{ terms})} =$$

$$\frac{n.n-1 \ldots \text{ to } 1}{\frac{1.2.3 \ldots \text{ to } n}{n.n-1\ (m \text{ terms})} \times 2n.2n-1\ (m \text{ terms})} =$$

$$\frac{n.n-1.n-2 \text{ (to } m \text{ terms)}}{2n.2n-1.2n-2 \text{ (to } m \text{ terms)}},$$ for the probability of getting m bowls at an end.

PROB. XXVI.

Any number (n) of things being given, a, b, c, d, &c. out of which m things are to be taken one by one. To find the probability of their being taken in the order they are written.

The number of permutations of n things taken m and m, is $n \times n-1 \times n-2$ &c. to m terms. But ſince the taking a, b, c, &c. in that order is but one caſe; therefore the probability of happening is

$$\frac{1}{n.n-1.n-2 \text{ (to } m \text{ terms)}}$$

EXAM.

EXAM.

What is the probability of taking 4 letters out of 6, in the order a, b, c, d.

Anſ. $\frac{1}{6.5.4.3} = \frac{1}{360}$ = the probability.

PROB. XXVII.

Any number (n) of things a, b, c, d, &c. being given; out of which m things are to be taken at once; to find the probability of taking thoſe propoſed.

The number of combinations of n things taken m at a time, is $\frac{n.n-1.n-2}{1.2.3}$ (to m terms); but the taking m things propoſed is but one caſe of theſe, therefore $\frac{1.2.3}{n.n-1.n-2}$ (to m terms) is the probability required.

EXAM.

What is the probability of taking the three d, e, f, out of the 6, a, b, c, d, e, f?

Anſ. Here $n = 6$, $\frac{1.2.3}{6.5.4} = \frac{1}{20}$ is the probability.

PROB. XXVIII.

What is the probability of throwing the ſix faces with 6 dice at a throw? And in how many throws may one undertake to do it?

Let a be the number of chances for happening, and b for failing. The number of permutations of 6 things, is the number of chances for happening, and that is $1 \times 2 \times 3 \times 4 \times 5 \times 6$. And the number

number of all the chances is 6^6; therefore the probability of happening is $\frac{1.2.3.4.5.6}{6.6.6.6.6.6} = \frac{4.5}{6.6.6.6} = \frac{5}{324}$. Here $a = 5$, and $b = 324 - 5 = 319$. Therefore (by Cor. 2. Prob. XI.) $\frac{7 \times 319}{5} = 44.8$; therefore in 44 or 45 throws, it is an equal chance for its happening or failing.

PROB. XXIX.

Three perſons A, B, C, *draw blindfold* 1 *counter at a time, out of a heap containing* 12 *counters,* 8 *black and* 4 *white ones. They draw in the order* A, B, C, A, B, *&c. till one of them gets a white counter, and then wins. What is the probability of each for winning?*

A's probability at firſt drawing is $\frac{4}{12}$, and his expectation of the ſtake $= \frac{4}{12} \times 1 = \frac{4}{12}$. This being taken out of the ſtake, there remains $1 - \frac{4}{12} = \frac{8}{12}$.

B's probability is $\frac{4}{11}$, and his expectation of the remaining ſtake is $\frac{8}{12} \times \frac{4}{11}$; then the remaining ſtake is $\frac{8}{12} - \frac{8.4}{12.11} = \frac{8.7}{12.11}$.

C's probability is $\frac{4}{10}$, and his expectation $= \frac{4.8.7}{12.11.10}$; and the remaining ſtake $= \frac{8.7}{12.11} - \frac{4.8.7}{10.12.11} = \frac{8.7.6}{12.11.10}$.

Then A's expectation again is $\frac{4.8.7.6}{12.11.10.9}$, and ſo on till the ſtake is gone.

Hence

Hence we have the ſeries $\frac{4}{12}$, $\frac{4.8}{12.11}$, $\frac{4.8.7}{12.11.10}$, $\frac{4.8.7.6}{12.11.10.9}$, &c. or $\frac{4}{12} + \frac{8}{11}P + \frac{7}{10}Q + \frac{6}{9}R + \frac{5}{8}S$ &c. in which we muſt take 9 terms, in which a white counter muſt neceſſarily be drawn. Then take for A, the firſt, fourth, ſeventh terms; for B the ſecond, fifth, eighth. For C the third, ſixth, &c. And the ſums will be the reſpective probabilities; that is, for A, $\frac{231}{495}$; for B, $\frac{159}{495}$; for C, $\frac{105}{495}$, which are as 77, 53, and 35.

And if there be more gameſters, the ſame ſeries will anſwer all.

PROB. XXX.

Three gameſters A, B, C, *throw in their turns a ball with* 4 *white faces and* 8 *black ones. What are the reſpective probabilities of throwing a white face firſt, and ſo to be the winner?*

Put $a = 4$, $b = 8$, $n = 12$, Then A's probability at the firſt throw, is $\frac{a}{n}$ = his expectation; this taken out of the ſtake 1, is $1 - \frac{a}{n} = \frac{n-a}{n} = \frac{b}{n}$.

Then, as the number of faces do not diminiſh, B's probability will be alſo $\frac{a}{n}$, and his expectation $= \frac{ab}{nn}$; and the remaining ſtake $= \frac{b}{n} - \frac{ab}{nn} = \frac{bb}{nn}$.

In

In like manner C's probability is $\frac{a}{n}$ and his expectation $\frac{abb}{n^3}$; and so on.

Therefore we shall have the series $\frac{a}{n} + \frac{ab}{nn} + \frac{abb}{n^3} + \frac{ab^3}{n^4} + \frac{ab^4}{n^5}$, &c. Then take every third term (beginning at the first, second, third) for the respective expectations of A, B, C. But these 3 series, will be in geometrical progression, which therefore are to one another as $\frac{a}{n}, \frac{ab}{nn}, \frac{abb}{n^3}$; or as the numbers 9, 6, 4.

And if there be more gamesters, their expectations will be as n, b, $\frac{bb}{n}$, $\frac{b^3}{nn}$, &c. respectively.

If the rule of the play be, that each man throws twice together; then take the 2 first terms for A, the 2 next for B, and so on.

PROB. XXXI.

There is a heap of counters, containing 4 white and 8 black ones. And A *engages, in taking 7 out of the heap, that 3 shall be white ones. To find his chance.*

If 12 counters in all give 4 white ones; then 7 counters will give $2\frac{1}{3}$ white ones. But as A was to take 3, his chance is $\frac{2\frac{1}{3}}{3}$ or $\frac{7}{9}$.

Otherwise.

All the cases in which 4 black ones out of 8 can be combined will be $\frac{8.7.6.5}{1.2.3.4} = 70$, and (in taking out

out 7) 4×70 will be the number of caſes, wherein 4 black ones and 3 white ones (out of 4) can be combined $= 280$; becauſe the combinations of 3 things out of 4 is 4.

But 4 white ones and 3 black ones may alſo happen; and the number of combinations of theſe is the ſame as the number of combinations of the three black ones alone, and that is $\frac{8.7.6}{1.2.3} = 56$; becauſe the number of combinations of 4 white ones out of 4 is but 1; and $56 \times 1 = 56$.

Therefore the number of caſes for happening is $280 + 56 = 336$. And the number of all the caſes that 7 out of 12 can be combined is $\frac{12.11.10.9.8.7.6}{1.2.3.4.5.6.7} = 792$; therefore A's probability of winning is $\frac{336}{792}$, and his chance of winning to the chance of loſing, as 336 to 456, or as 14 to 19.

PROB. XXXII. (L. Diary, 1756.)

Suppoſe 12 half-pence thrown up, and thoſe that come up heads to be taken away; and the remaining ones thrown up again, and ſo on; till at laſt all the half-pence are thrown up heads. In how many throws is it equally probable for happening or failing.

Since one time with another half will be taken away every throw; we muſt proceed to biſect the number till we come at 1, then the number of biſections + 1 = number of throws required. Thus 12, 6 (1), 3 (2), $1\frac{1}{2}$ (3), $\frac{3}{4}$ (4); whence between 4 and 5 throws are required.

Therefore take a ſeries of terms in geometrical progreſſion as 1 to 2, viz. 1, 2, 4, 8, 16, &c. and take that term which is neareſt the number given, and

and its place in the ſeries will ſhew the number of throws.

PROB. XXXIII.

In a lottery where there are 13333 *blanks to a prize; what is the probability of geting a prize in* 8000 *tickets?*

The probability is $\frac{8000}{13333} = \frac{8}{13}$ nearly.

PROB. XXXIV.

In a lottery containing a great number of tickets, having 13333 *blanks to a prize. How many tickets muſt be taken to make it an equal chance for getting a prize.*

Here $a = 1$, $b = 13333$, and $t = 1$, then (by Prob. XV.) $\overline{t - .3} \times \frac{b}{a} + t - 1 = .7 \times \frac{b}{a} = .7 \times 13333 = 9333$ the number of tickets.

SCHOL.

It may be obſerved, that in many of theſe problems, to avoid more intricate methods of calculation, I have contented myſelf with a more lax method of calculating, by which I only approach near the truth. For ſome problems require ſuch laborious computations, as no body can be able to go through; and which is worſe, they often only ſerve for particular caſes. And theſe I have endeavoured to avoid, upon account of the uncertainty of chance, as not requiring any ſuch ſtrictneſs. For in a ſingle trial hardly any thing ever anſwers calculation; yet being often repeated it comes nearer and nearer.

Hence

Hence we may obſerve, that altho' we may by an equality of chance, contend, that a ſingle event will happen in a certain number of trials; yet the ſame will not happen twice in double that number, nor thrice in tripple that number, &c. But the time will grow longer and longer; and therefore at long run, theſe events will not happen oftener than in the proportion of the chances they have for happening at firſt; but will keep conſtantly to that ratio.

Hence we may ſee, that in a great length of time, chance very little diſturbs ſuch events, as were deſigned to happen according to ſome determinate law; for however irregular theſe events may be at firſt, or in a few trials, yet in time they continually converge to a certain regular proportion; all irregularities continually correcting one another, ſo that theſe events continually tend to ſome determined rule. Whence follows this wonderful concluſion, that regularity ariſes out of irregularity, and order out of diſorder.

And as we find that the happening of all events continually approach to ſome rule or law, according as experiments and obſervations increaſe. So on the other hand, when we find by multiplicity of obſervations, that theſe effects of chance ultimately converge to ſome determined law; we conclude that this is the very law, according to which theſe events were originally deſigned to happen.

And hence we may conclude, that all ſuch laws, by which the various effects and productions of chance are regulated, are the effects of intelligence and deſign; and were inſtituted by ſome powerful agent, and intended to ſerve for ſome wiſe and uſeful purpoſes, which are neceſſary for preſerving the order and œconomy of the univerſe. So that the great Author of Nature can make even Chance execute his deſigns.

ART.

ART. II.

ANNUITIES UPON LIVES.

THIS is a branch of the Doctrine of Chances; and beſides, depends on the laws of mortality of mankind. The former has been briefly treated on; and the latter depends entirely upon obſervation and experience. To proceed regularly in this affair, the rate of people dying every year muſt be extracted from the bills of mortality of ſome populous place or places, and put regularly into a table, which muſt ſtand as the baſis of all calculations to be made for the purpoſe of annuities on lives.

NOTATION.

J. AB, ſignifies the joint lives of the two perſons A, B. And J. ABC, the joint-lives of the three perſons A, B, C.

L. AB, ſignifies the longeſt of the two lives A, B. And L. ABC, the longeſt of the three lives A, B, C, &c.

PROP. I.

Suppoſing a number of children born in any year; the decrements of life will be nearly in arithmetic progreſſion; or an equal number yearly will die every year, till they be all dead, which will be about 87 *or* 88 *years after.*

This appears from the following table, which is made by comparing four or five different tables with

one another; which tables were each of them made from the bills of mortality of ſo many different places. So that this here being a mean among all theſe, is likely to be more true, and more generally uſeful than any one of the reſt.

From this table it appears, that from the birth to the age of 7 or 8, mankind die faſter; for out of 1000 that are born, 151 die in the firſt year, 108 in the ſecond, 52 in the third, &c. But from the age of eight to twenty-two the feweſt die, which is only 5 or 6 in a year; and afterwards 7, 8, or 9, till 80. And if 7 or 8 was to continue to die yearly, all the lives would then terminate about 88.

Now ſince theſe numbers, which are the differences of the people living each year, are ſo ſmall in reſpect of the numbers themſelves; and continue equal for large intervals of time; therefore the number of perſons living year after year, for theſe intervals, will then be exactly in arithmetic progreſſion; and through the whole from 8 to 88 may be accounted ſo without any ſenſible error.

A Table

A Table of the people living at all ages; from the Bills of Mortality.

Age. born.	livin. 1000	dif.	ag.	liv.	dif.	ag.	liv.	dif.	ag.	liv.	dif.
1	849	151	26	500	7	51	307	9	76	88	9
2	741	108	27	493	8	52	298	8	77	79	9
3	689	52	28	485	7	53	290	9	78	70	8
4	661	28	29	478	8	54	281	8	79	62	8
5	642	19	30	470	8	55	273	8	80	54	7
6	627	15	31	462	8	56	265	9	81	47	7
7	615	12	32	454	8	57	256	8	82	40	6
8	605	10	33	446	7	58	248	9	83	34	5
9	597	8	34	439	8	59	239	9	84	29	5
10	590	7	35	431	8	60	230	8	85	24	5
11	584	6	36	423	7	61	222	8	86	19	5
12	579	5	37	416	8	62	214	9	87	14	5
13	574	5	38	408	7	63	205	8	88	9	5
14	569	5	39	401	8	64	197	9	89	4	4
15	565	4	40	393	8	65	188	8	90	0	
16	560	5	41	385	8	66	180	9	31710		
17	555	5	42	377	7	67	171	9	sum of all.		
18	550	5	43	370	8	68	162	10			
19	545	5	44	362	8	69	152	9			
20	539	6	45	354	7	70	143	9			
21	533	6	46	347	8	71	134	10			
22	527	6	47	339	8	72	124	9			
23	521	7	48	331	8	73	115	9			
24	514	7	49	323	8	74	106	9			
25	507	7	50	315	8	75	97	9			

This table needs little explanation; for the firſt col. is the age; the ſecond is the number of perſons (out of 1000) living at that age; the third ſhews their differences, or the perſons that die in a year, at that age.

Cor. *If a right line be drawn, and divided into* 88 *equal parts, and at the ſeveral points of diviſion, perpendiculars*

pendiculars be erected proportional to the numbers in the table, and a line be drawn through the tops thereof; this figure will represent the state of mankind in all ages; and will be very nearly a triangle.

If there be taken 78 children, each 10 years old; then one will die every year, till they be all extinct.

PROP. II. PROB.

To shew the use of the foregoing table, in matters relating to human life.

1. *To find how many people there are of any age, compared with the whole number of people in a nation.* Suppose it be required to know how many persons there are from 21 to 25 years old (inclusive). Add up these 5 numbers, from 21 to 25, and the sum is 2602. And the whole number by the table is 31710. Therefore $\frac{2602}{31710} \times$ number of inhabitants, shews how many there are of these ages.

2. *To find how many years, it is an even wager, whether a person of a given age shall live or die; as suppose his age be* 38. Take the number against 38, which is 408, and half it, which will be 204; then seek it in the table, which will be at 63. Therefore it is an even wager that he lives till 63, that is 25 years.

3. *To find the difference of insurance upon different lives; as between* 30 *and* 50. Here it is 470 to 8, or 59 to 1, that a man of 30 dies not in a year; and only 315 to 8, or 39 to 1, for a man of 50. The price of insurance then ought to be as 59 to 39, or as 3 to 2 nearly.

4. *To find the probability of a person of a given age a, living any number of years t.* Let N = number of persons living at the age *a*, L = persons living

living at the age $a + t$, then $\frac{L}{N}$ = probability of his living t years. And $\frac{N - L}{N}$ = probability of his being dead in t years.

5. *To find the chances of two lives*, A *and* B, *whose ages are a and b.* Thus by the table, let t = any time, and

N = number of persons living at the age a.
n = persons living at the age b.
L = persons living at the age $a + t$.
l = persons living at the age $b + t$.
$D = N - L$ = persons dead in t years (after a).
$d = n - l$ = persons dead in t years (after b).

Then,

$\frac{Ll}{Nn}$ = probability of both A and B living t years.

$\frac{Dd}{Nn}$ = probability of both being dead in t years.

$1 - \frac{Dd}{Nn}$, or $\frac{Nn - Dd}{Nn}$ = probability of one of them living t years.

$\frac{Ld}{Nn}$ = probability of A being alive and B dead in t years.

6. *To find the chances for three lives*, A, B, C, *whose ages are a, b, c.*

Denoting the numbers as in Art. 5. And moreover putting p = persons living at the age c, m = persons living at the age $c + t$, $f = p - m$ = persons dead in t years after c.

Then,

$\frac{Llm}{Nnp}$ = probability of all three living t years.

$\frac{Ddf}{Nnp}$ = probability of all being dead in t years.

$1 - \frac{Ddf}{Nnp}$, or $\frac{Nnp - Ddf}{Nnp}$ = probability of ſome of them living t years.

$\frac{Ldf}{Nnp}$ = probability of A being alive and the reſt dead in t years.

$\frac{Llf}{Nnp}$ = probability of A and B being alive, and C dead in t years.

7. *To find the value of a ſum of money, due t years hence, for one or more lives.* It is plain the purchaſer ought only to pay ſuch a part of the preſent value, as there are chances for the perſons living. Therefore (by Art. 4, 5, or 6,) find the probability of the perſon or perſons living t years hence, and multiply this by the preſent worth of 1*l.* and that product by the ſum given, for the value required. As for example, to find the value of 100*l.* due 12 years hence; upon the longeſt of two lives, aged 40 and 50, at 4 *p.* cent. Here (Art. 5.) $t = 12$. $N = 393$, $n = 315$, $D = 393 - 298 = 95$, $d = 315 - 214 = 101$. And preſent value of 1*l.* = .6246; and the probability $= \frac{Nn - Dd}{Nn} = \frac{114200}{123795} = .9225$. Then $.6246 \times .9225 \times 100 = 57^{11}.62$, the preſent value.

8. *To find the value of an annuity for one or more lives, or to continue t years, if the perſons live as long.* By Art. 7. find the preſent value of the annual rent for 1 year, for 2 years, for 3 years, and ſo on till you get t years; then the ſum of all theſe is the value required for t years. But if t is not given, this proceſs is to be continued, till the terms be 0, or ſo ſmall as to be inconſiderable, which will be at the extremity of old age. This often requires a la-

a laborious calculation, and therefore ſhorter methods will be ſhewn after.

9. *Any number of equal ages being given; to find the value of the longeſt life.* Take the number in the table againſt the common age given, and divide it by the number of lives. Seek the quotient (or the next leſs) in the table, and take the age againſt it, from which ſubtract the given age; the remainder ſhews how long an annuity certain is to continue, which is equal in value to the longeſt life.

Suppoſe 5 perſons, of 38 years old. Againſt 38 is 408, and $\frac{408}{5} = 81\frac{1}{2}$, and againſt 81 is 77; whence $77 - 38 = 39$ years continuance for the value of the longeſt life.

10. *To find the value of the reverſion, after one or more lives.* Find the value of the annuity for the life or lives propoſed, by Art. 8; and ſubtract it from the value of the perpetuity.

11. *To find the complement of life for any age by this table, or any ſuch table made from the bills of mortality.*

Find the difference between the number of that age, and the number of 5 years difference. Then as that difference, to 5 : : ſo is the number of either age, to its complement. Or inſtead of 5 years, one may take any other difference of years.

Exam. Take the ages 5 and 10; then $642 - 590 = 52$, and 52 : 5 : : 590 : 57, the comp. life for 10. This makes the extremity of old age only 67, ſo that people die very faſt at this age.

Again, for the ages 20 and 25, $539 - 507 = 32$, and 32 : 5 : : 539 : 84 the comp. of life for 20; which makes the extremity of old age 104, ſo that now people die very ſlow.

Laſtly, take the ages 48 and 53, $331 - 290 = 41$, and 41 : 5 : : 290 : 35, the comp. life for 53,

and $35 + 53 = 88$, the extremity of old age at this rate.

If this be done for every 5 or every 10 ages from 8 to 88, by this table; the mean quantity for the extremity of old age, will come out 94.

12. *Any number of equal lives being given; to find the time in which any number of them will fail.*

Let p be the number of all the lives, and q the number of theſe to die. Take the number againſt the given age in the table; and take $\frac{p-q}{p} \times$ that number, and ſeek it in the table; againſt which is the age that q of them will be dead. The difference of theſe ages ſhews the time.

Exam. Suppoſe 12 lives of 40, and 5 to fail. Againſt 40 is 393, and $\frac{7}{12} \times 393 = 229$, againſt which is 60 nearly; then $60 - 40 = 20$ years, the time required.

13. *To find the probability of a perſon of a given age (a) living t years; and afterwards dying in v years.*

Find the number p in the table againſt his age a, alſo the number q againſt $a + t$, and the number r againſt $a + t + v$; then the probability required is $= \frac{q-r}{p}$.

Suppoſe the age $a = 48$, $t = 13$, $v = 5$; then $p = 331$, $q = 222$, $r = 180$, then the probability $= \frac{222 - 180}{331} = \frac{42}{331}$.

PROP. III. Prob.

To find the expectation of a ſingle life, or the time a perſon of a given age, may reaſonably expect to live.

This

This has been reſolved by the table in Art. 2. of the laſt Prop. I ſhall here reſolve it by calculation, ſuppoſing the decrements of life to be equal. Let x be the time ſought, a the perſons age, and $88 - a = n =$ the complement of life; then the probability of living x years is $= \frac{x}{n}$, and ſuppoſing it an equal chance for living or dying in that time, we have $\frac{x}{n} = \frac{1}{2}$, whence $x = \frac{1}{2} n$, or half the complement of life.

Exam. Suppoſe $a = 38$, then $n = 50$, and $\frac{1}{2} n = 25$, his expectation of life.

PROP. IV. PROB.

The ages of two perſons being given, to find the expectation of their joint lives; or how long they may reaſonably expect to live together.

Let n, m, be the complements of life, n being the leaſt. Then the probability of their joint continuance for 1, 2, 3, &c. years, will be $\frac{m-1}{m} \times \frac{n-1}{n} + \frac{m-2}{m} \times \frac{n-2}{n} + \frac{m-3}{m} \times \frac{n-3}{n}$ &c. to n terms; that is,

$\frac{1}{mn} \times$ into $\quad mn - 1 \times \overline{m+n} + 1$

$+ mn - 2 \times \overline{m+n} + 4$
$+ mn - 3 \times \overline{m+n} + 9$
$+ mn - 4 \times \overline{m+n} + 16$
&c. to n terms.

But

But the ſum of all the firſt column $= n \times mn$,

And the ſum of the ſecond column $= -\frac{n+1}{2} \times \overline{m+n}$

And the ſum of the third column $= n \times \frac{n+1}{2} \times \frac{2n+1}{3}$, by the differential method.

And the whole ſum $= \frac{mn^2}{2} - \frac{mn}{2} - \frac{n^3-n}{6}$; and dividing by mn, the expectation will be $= \frac{n-1}{2} - \frac{nn-1}{6m}$.

But this may otherwiſe be ſolved by finding n terms of the ſeries $\frac{mn}{mn} + \frac{m-1}{m} \times \frac{n-1}{n} + \frac{m-2}{m} \times \frac{n-2}{n}$ &c. that is,

$\frac{1}{mn} \times$ into $mn - 0 \times \overline{m+n} + 0$

$+ mn - 1 \times \overline{m+n} + 1$

$+ mn - 2 \times \overline{m+n} + 4$

$+ mn - 3 \times \overline{m+n} + 9$

&c.

And the ſum found as the foregoing, and divided by mn, will be $\frac{1}{2}n - \frac{nn}{6m} + \frac{1}{2} + \frac{1}{6m}$, that is, $\frac{n+1}{2} - \frac{nn-1}{6m}$, for the expectation. And taking the mean between this and the former, we ſhall have $\frac{n}{2} - \frac{nn-1}{6m}$, for the true expectation. And neglecting $\frac{1}{6m}$ as being very ſmall, the value

of

of the expectation of the two joint lives will be $\frac{n}{2} - \frac{nn}{6m}$.

Cor. *If both the lives are equal, the expectation will be* $\frac{1}{3}n$, *or* $\frac{1}{3}$ *their common complement.*

PROP. V. PROB.

The ages of three persons being given, to find the expectation of their joint lives; or the time they may expect to live together.

Let n, m, p, be their complements of life, and n the least. Then the probabilities of continuing together for 1, 2, &c. years will be $\frac{m-1}{m} \times \frac{n-1}{n} \times \frac{p-1}{p} + \frac{m-2}{m} \times \frac{n-2}{n} \times \frac{p-2}{p} +$ &c. to n terms; that is, $\frac{1}{mnp} \times$ into

$$\begin{aligned} &mnp - 1 \times R + 1 \times S - 1 \\ &+ mnp - 2 \times R + 4\,S - 8 \\ &+ mnp - 3\,R + 9\,S - 27 \\ &+ mnp - 4\,R + 16\,S - 64 \end{aligned}$$

&c. where $R = mn + mp + np$. $S = m + n + p$.

Then by the differential method,

The sum of the first column $= n \times mnp$.

The second column $= -\frac{n+1}{2} \times nR$.

The third column $= n \cdot \frac{n+1}{2} \cdot \frac{2n+1}{3} S$.

The fourth column $= -\frac{n^4 + 2n^3 + n^2}{4}$.

And the sum of all, divided by mnp, will be $\frac{n-1}{2} - \frac{n^2-1}{6p} - \frac{n^2-1}{6m} + \frac{n^2-1}{12mp} n$.

But

But if it be ſolved by finding n terms of the ſeries,

$$\begin{array}{l} mnp - 0 \\ + mnp - 1\,R + 1\,S - 1. \\ + mnp - 2\,R + 4\,S - 8. \\ + mnp - 3\,R + 9\,S - 27. \\ \quad \&c. \end{array}$$

Then the reſult will be $\frac{n+1}{2} - \frac{n^2-1}{6p} - \frac{n^2-1}{6m} + \frac{n^2-1}{12mp}\,n.$

And the mean between the two is $\frac{n}{2} - \frac{n^2-1}{6p} - \frac{n^2-1}{6m} + \frac{n^2-1}{12mp}\,n$, for the expectation of 3 joint lives. Or leaving out ſome very ſmall quantities, the expectation will be $\frac{n}{2} - \frac{n^2}{6m} - \frac{n^2}{6p} + \frac{n^3}{12mp}$.

Cor. *If all the lives are equal, then the expectation will be* $\frac{1}{4}\,n$.

SCHOLIUM.

After the ſame manner the expectation of 4 joint lives would be $\frac{1}{2}\,n - \frac{nn}{6m} - \frac{nn}{6p} - \frac{nn}{6q} + \frac{n^3}{12mp} + \frac{n^3}{12mq} + \frac{n^3}{12pq} - \frac{n^4}{20mpq}$, where q is the complement of the 4th life.

And if all the lives are equal, the expectation $= \frac{1}{5}\,n$. And in general, if r be the number of equal lives, then the expectation of the joint lives will be $\frac{n}{r+1}$.

PROP. VI. PROB.

To find the expectation of the longeſt of two or more lives whoſe ages are given.

Let

Let m, n, p, q, &c. be the complements of the ſeveral ages from the youngeſt to the oldeſt; that is, m being the greateſt, n the next, and ſo on in order. Then $\frac{1}{2}m + \frac{nn}{2.3m} + \frac{p^3}{3.4mn} + \frac{q^4}{4.5mnp}$ + &c. (continued to as many terms as there are perſons) will be the expectation of the longeſt life.

For in one life the expectation will be $\frac{1}{2}m$; and in two lives the ſecond perſon has a chance to outlive the firſt, which muſt be added to the former, and this chance is $\frac{nn}{6m}$; being the ſame chance as the value of the ſhorteſt life was diminiſhed, for two joint lives (Prop. IV.)

And in three lives, the third has a chance to outlive the other two, which is $\frac{p^3}{12mn}$, this muſt be added to the former.

And in four lines, the fourth has a chance to outlive the other three, which is alſo to be added, and is $\frac{q^4}{20mnp}$; and ſo on.

Cor. 1. *If all the lives are equal, and r their number; then* $\frac{rn}{r+1}$ = *expectation of the longeſt liver.*

For in two lives the expectation $= \frac{1}{2}n + \frac{1}{6}n = \frac{4}{6}n = \frac{2}{3}n$.

And in three lives, it is $= \frac{1}{2}n + \frac{1}{6}n + \frac{1}{12}n = \frac{9}{12}n = \frac{3}{4}n$.

And in four lives it is $= \frac{1}{2}n + \frac{1}{6}n + \frac{1}{12}n + \frac{1}{20}n = \frac{3}{4}n + \frac{1}{20}n = \frac{16}{20}n = \frac{4}{5}n$, and ſo on.

Cor.

Cor. 2. *If there be r equal lives; the time wherein g of them will fail will be* $\frac{g}{r+1}n$.

For if $g = r$, then the time $= \frac{r}{r+1}n$, by Cor. 1; and if $g = 1$, the time $= \frac{1}{r+1}n$, by Schol. Prop. V. and there is the same reason for the rest.

PROP. VII.

The joint lives of two persons + the longest life of the same two persons is = sum of their single lives.

Let the two persons be A, B. Their joint lives is the same as the shortest life; and which ever dies first (as A), being the shorter life; the other (B that remains) is the longer life; so that one always makes up the other; and their sum is the life of A + the life of B. Therefore the joint lives of A, B + longest life of A, B = life of A + life of B.

Cor. *The longest of two lives is equal to the sum of the single lives — the joint lives.*

PROP. VIII.

If ABCFGH *be several people, and* Z *a life equal to the longest life of* ABC; *and* Y *a life equal to the longest of the rest* FGH. *Then the value of the longest life of* ZY *is = value of the longest life of the whole number* ABCFGH.

For the longest life of ABCFGH is the very same as the longest life of the two parts ABC and FGH, or of the two Z and Y. Consequently the present

preſent values are equal; that is, value of *l.* ABCFGH = value of *l.* ZY. Where *l.* ſignifies longeſt life.

Cor. 1. *If* Z = *longeſt life of* AB; *then longeſt life of* ZF = *longeſt life of* ABF. *Or l.* ZF = *l.* ABF.

Cor. 2. *If* Z = *l.* ABC; *then l.* ZF = *l.* ABCF.

Cor. 3. *If* Z = *l.* AB; Y = *l.* FG; *then l.* ZY = *l.* ABFG, &c.

PROP. IX.

If ABCFGH *&c. be ſeveral lives, and* Z *a life equal to the joint lives of* ABC; *and* Y *a life equal to the joint lives of the reſt* FGH. *Then the value of the joint lives of* ZY = *value of the joint lives of all* ABCFGH. *Or j.* ZY = *j.* ABCFGH.

For the joint lives of any number of people, is the ſhorteſt life among them. And the ſhorteſt life of ABCFGH is the ſame thing as the ſhorteſt life of the two parcels ABC and FGH, or of the two lives Z and Y; that is, the joint lives of ABCFGH = joint lives of ZY, and therefore their preſent values are equal.

Cor. 1. *If* Z = *joint lives of* AB; *then the joint lives of* ZF = *joint lives of* ABF; *or j.* ZF = *j.* ABF.

Cor. 2. *If* Z = *joint lives of* ABC; *then j.* ZF = *j.* ABCF.

Cor. 3. *If* Z = *joint lives of* AB, Y = *joint lives of* FG, *then j.* ZY = *j.* ABFG.
And ſo of others.

Cor. 4. *If* A, B, C, *be three lives, and* Z = *l.* AB; *then j.* $\overline{CZ}$ = *j.* AC + *j.* BC − *j.* ABC.

For

For $j.\ \overline{CZ} = j.\ C$ and $l.\ AB = j.\ \overline{C \text{ and } A+B} - j.\ AB$ (Prop. VII.) $= j.\ AC + j.\ BC - j.\ ABC$.

Cor. 5. *If* A, B, C, *be three lives, and* $Z = j.\ AB$; *then* $l.\ \overline{ZC} = l.\ AC + l.\ BC - l.\ ABC$.

For $l.\ \overline{CZ} = l.\ C$ and $j.\ AB = l.\ \overline{C \text{ and } A+B} - l.\ AB$ (Prop. VII.) $= l.\ AC + l.\ BC - l.\ ABC$. And ſo of others.

PROP. X.

The value of an annuity for the longeſt life of any number of perſons ABCFGH — *the value of the longeſt life of any part of them* ABC = *value of the reverſion of the longeſt life of the reſt* FGH *after the longeſt life of* ABC. *Or* $l.$ ABCFGH — $l.$ ABC = *rever.* $l.$ FGH *after* $l.$ ABC, *where* $l.$ *ſignifies the longeſt life.*

For the intereſt or advantage that ABC have in the annuity, and alſo what FGH have in it, is equal to the whole advantage that they all (ABCFGH) have in it, and that is for the longeſt life of ABCFGH. Therefore

From the value of the longeſt life of ABCFGH,
Subtract the value of the longeſt life of ABC,
The remainder is the value of the reverſion of the longeſt of the lives of FGH after thoſe of ABC.

That is, $l.$ ABCFGH — $l.$ ABC = rev. $l.$ FGH after $l.$ ABC.

Cor. 1. *The value of* $l.$ AF — *life of* A = *revers.* F *after* A.

Cor. 2. *Value of* $l.$ AFG — a = *rever.* $l.$ FG *after* A, *where* a = *life of* A.

Cor. 3. *Value of* $l.$ ABF — $l.$ AB = *rev.* F *after* $l.$ AB.

&c.

PROP.

PROP. XI.

The value of an annuity for the joint lives of FGH *— the value of the joint lives of* ABCFGH = *value of the reverſion of the joint lives of* FGH *after the joint lives of the reſt* ABC. *Or j.* FGH *— j.* ABCFGH = *rev. j.* FGH *after j.* ABC.

Put Y = *j.* ABC, Z = *j.* FGH; then *j.* YZ = *j.* ABCFGH. But *j.* YZ = is the ſame as the ſhorter life, which here is Y becauſe Z (*j.* FGH) is ſuppoſed the value of the whole. Therefore *j.* ZY being the life of Y, Z — *j.* ZY = life of Z after Y; that is, (reſtoring the letters) *j.* FGH — *j.* ABCFGH = rev. *j.* FGH after *j.* ABC.

Cor. 1. *Joint lives of* FG — *joint lives of* AFG = *rever. j.* FG *after* A.

Cor. 2. *J.* FGH — *j.* AFGH = *rever. j.* FGH *after* A.

Cor. 3. F — *j.* ABF = *rever.* F *after j.* AB.

Cor. 4. F — *j.* ABCF = *rev.* F *after j.* ABC.

Cor. 5. *j.* FG — *j.* ABFG = *rev. j.* FG *after j.* AB.

SCHOL.

By help of the foregoing propoſitions, the value of annuities for joint lives, or for longer lives, when they are many, or for reverſions of ſuch lives, may be far eaſier found, than by the laborious and intricate methods hitherto uſed for theſe purpoſes. And indeed many problems cannot be ſolved by that way, which are eaſily managed by this new and ſhort method.

PROP. XII. PROB.

Supposing the decrements of life to be equal; to find the present value of 1*l. due t years hence, if a person of a given age lives so long.*

Let n be the complement of the person's age; then since the money is to be valued according to the chance there is for the person's living, and that chance is $\frac{n-t}{n}$; therefore $\frac{n-t}{n}$ is to be multiplied into the present worth of 1*l.* due t years hence, which is $\frac{1}{R^t}$ where R is the amount of 1*l.* for a year.

Therefore the present value will be $\frac{n-t}{n} \times \frac{1}{R^t}$ or $\frac{n-t}{nR^t}$.

Cor. 1. *If* S *is any sum of money due t years hence for a life; its value* $= \frac{n-t}{nR^t}$ S.

Cor. 2. *If several sums of money* S, T, U, *be due on a life, t, v, w, years hence. The present value* $= \frac{n-t}{nR^t}$ S $+ \frac{n-v}{nR^v}$ T $+ \frac{n-w}{nR^w}$ V.

EXAM.

What is the present value of 1*l.* due 10 years hence, if a person 40 years old lives so long; at 4 *per* cent.

By Tab. I. $\frac{1}{R^t} = .6755$, $n = 88 - 40 = 48$; and $t = 10$, and $n - t = 38$, then $\frac{38}{88} \times .6755 =$ $.292 = 5 - 10$.
l. *s.* *d.*

PROP.

PROP. XIII. PROB.

To find the present value of an annuity of 1l. a year, for t years; if a person of a given age lives so long. But if he does not, the annuity is to cease at his death.

Let n be the complement of the given age; and R = amount of 1*l.* for a year, r = its interest. Suppose the several payments not to be at 1, 2, 3, &c. years distance, but sooner by the time v; then $1-v$, $2-v$, $3-v$, &c. will be the times of the payments. Therefore (by Cor. 2. Prop. XII.) the present value of the annuity will be $\frac{n+v-1}{nR^{1-v}}$ $+\frac{n+v-2}{nR^{2-v}}+\frac{n+v-3}{nR^{3-v}}$ &c. to t terms; for in t years the annuity is supposed to be ended. And this will be divided into the two series,

$$R^v \times \frac{n+v}{n} \times \frac{1}{R}+\frac{1}{R^2}+\frac{1}{R^3} \text{ &c. to } t \text{ terms.}$$

$$-\frac{R^v}{n} \times \frac{1}{R}+\frac{2}{R^2}+\frac{3}{R^3} \text{ &c. to } t \text{ terms.}$$

The former series $= \frac{n+v}{n} R^v \times \frac{1-\frac{1}{R^t}}{r}$ (Prop. XXVI. proportion) and the latter $= -\frac{R^v}{n} \times : \frac{R}{rr}-\frac{t}{rR^t}-\frac{R}{rrR^t}$: (increments, Prop. XXX.) put $p=\frac{1}{R^t}$, and the sum of the two series $= R^v \times \frac{n+v}{n} \times \frac{1-p}{r}-\frac{R}{nrr}+\frac{tp}{nr}+\frac{Rp}{nrr} : =$

$\frac{\overline{n+v-n+v.p+tp}}{nr} R^v - \frac{1-p}{nrr} R^{1+v} =$ value of the annuity, where $p = \frac{1}{R^t}$.

Cor. 1. *If the payments begin at a year's end,* $v=0$, *and then the value* $= \frac{\overline{1-p.n+pt}}{nr} - \frac{1-p}{nrr} R.$

Cor. 2. *If the payments begin immediately, add* 1 *to the value found by Cor.* 1.

EXAMP.

Suppoſe the age be 56, and the annuity be for 16 years, at 5 *per* cent. Here (by Tab. I.) $p = .458$, and $1-p = .542$, $n = 32$, $t = 16$. Then by Cor. 1.

$$\frac{\overline{1-p.n+pt}}{nr} = 15.42$$
$$\frac{1-p}{nrr} R = 7.11$$
$$8.31$$ the value req.

If the payments begin immediately, the value $= 9.31$.

PROP. XIV. PROB.

To find the preſent value of an annuity of 1*l. for the life of a given age.*

1 *Way.*

I ſhall ſuppoſe, as in the laſt Prop. that the firſt payment is not exactly at a year's diſtance, but ſooner by the time v; and that the decrements of life are equal, and that the extremity of old age is 88,

88, as agreeable to my general table. Now the ſeveral rents being ſo many ſums of money payable at different times, all at a year's diſtance; therefore proceeding as in the laſt Prop. (according to Cor. 2. Prop. XII.) the preſent value of the annuity will be $\frac{n+v-1}{n\,R^{1-v}} + \frac{n+v-2}{n\,R^{2-v}} + \frac{n+v-3}{n\,R^{3-v}}$ &c. to n terms; for in n years the life will be extinct. This conſiſting of two ſeries, the ſum will be found, as in the laſt Prop. to be $R_v \times$ into $\frac{n+v}{n} \times \frac{1-p}{r} - \frac{R}{nrr} + \frac{np}{nr} + \frac{Rp}{nrr}$: that is, the value of the annuity $= \frac{n+v-vp}{nr} R^v - \frac{1-p}{nrr} R^{1+v}$; where $p = \frac{1}{R^n}$ the preſent value of 1*l.* annuity due n years hence.

Cor. 1. *If* P *is the preſent worth of* 1*l. annuity to continue n years certain; then the value of the annuity for the life whoſe complement is n, is* $= \frac{n - \overline{R - vr.P}}{nr} R^v$; *or for* R^v *take* $\overline{1 + rv}$.

For (by Prob. XXXIV, *Algebra*) $P = \frac{R^n - 1}{rR^n}$ whence $R^n = \frac{1}{1 - rP}$, and $\frac{1}{R^n} = 1 - rP = p$, which put for p, will give the rule as above.

Cor. 2. *If the payments begin at the year's end*; $v = 0$, *and the value* $= \frac{1}{r} - \frac{1-p}{nrr} R = \frac{n - RP}{nr}$.

Cor. 3. *If the payments begin immediately, then* $v = 1$, *and the value* $= \frac{nr - 1 + p}{nrr} R = \frac{n-P}{nr} R$. *Or elſe add* 1 *to the value, by Cor.* 2.

EXAMP.

Suppoſe a life of 40, intereſt 4 *per* cent. $n = 48$, $R = 1.04$, $r = .04$; $p = .152$. $P = 21.195$. Then by Cor. 2.

$$\frac{1}{r} = 25.00, \qquad \text{Or,}$$

$$\frac{1-p}{nrr} R = 11.49. \qquad \frac{n-RP}{nr} = 13.52$$

$$\text{Annuity} = \overline{13.51}$$

Or if the annuity commence at preſent $13.52 + 1 = 14.52$, the value. Otherwiſe by Cor. 3.

$$\frac{nr-1+p}{nrr} R = \frac{1.115}{.0768} = 14.52$$

$$\text{Or } \frac{n-P}{nr} R = \frac{27.877}{1.92} = 14.52$$

2. *Otherwiſe thus.*

Let the complement of life be n, R the amount of 1*l.* for a year, r its intereſt, then the probability of living 1, 2, 3, &c. years is $\frac{n-1}{n}$, $\frac{n-2}{n}$, $\frac{n-3}{n}$, &c. and the preſent value of all the rents will be $\frac{n-1}{nR} + \frac{n-2}{nR^2} + \frac{n-3}{nR^3}$ &c. for n terms; but as $R = 1 + r$, $R^2 = 1 + 2r$, $R^3 = 1 + 3r$, &c. very near, we ſhall have the preſent value aforeſaid, that is, the value of the annuity $= \frac{n-1}{n.1+r} + \frac{n-2}{n.1+2r} + \frac{n-3}{n.1+3r}$, &c. to n terms. And to find the ſum S by the method of increments,

ments, here s, or the $\overline{z+1}^{th}$ term $= \frac{n - \overline{z+1}}{n \times 1 + \overline{z+1}.r}$; put $v = 1 + \overline{z+1}.r$, then $z + 1 = \frac{v-1}{r}$, and $s = \frac{n - \frac{v-1}{r}}{nv} = \frac{nr+1-v}{nrv} = \frac{nr+1}{nr} \times \frac{1}{v} - \frac{v}{nrv}$. But by Ex. 12, Prop. XIII. Increments, the integral of $\frac{1}{v} = \frac{h.\log.v}{v} - \frac{1}{2v} - \frac{v}{12vv}$ &c.

But $\dot{v} = r\dot{z} = r$, becauſe $\dot{z} = 1$; therefore int. of $\frac{1}{v} = \frac{h.\log.v}{r} - \frac{1}{2v} - \frac{r}{12vv}$ &c. Alſo (by form 1, or Cor. Ex. 10,) the integ. of $\frac{v}{nrv}$ or $\frac{1}{nr} = \frac{v}{nrv} = \frac{v}{nrr}$. Therefore int. $\frac{nr+1}{nrv} - \frac{1}{nr} = \frac{nr+1}{nr} \times$ into $\frac{h.\log.v}{r} - \frac{1}{2v} - \frac{r}{12vv} : -\frac{v}{nrr}$. But when $s = 0$, $z = 0$, and $v = 1 + r = R$, therefore s or $0 = \frac{nr+1}{nr} \times \frac{h.\log.R}{r} - \frac{1}{2R} - \frac{r}{12vv} : -\frac{R}{nrr}$, which ſubtracted gives $s = \frac{nr+1}{nr} \times \frac{h.\log.v}{r} - \frac{1}{2v} - \frac{r}{12vv} : -\frac{v}{nrr} + \frac{nr+1}{nr} \times -\frac{h.\log.R}{r} + \frac{1}{2R} + \frac{r}{12RR} : +\frac{R}{nrr}$. But $\frac{v-R}{nrr} = \frac{zr}{nrr} = \frac{z}{nr}$. Therefore put-

ting $Q = \frac{nr+1}{nr} \times \frac{b.\log.R}{r} - \frac{1}{2R} - \frac{r}{12RR}$: $+ \frac{z}{nr}$; the value of the annuity $= \frac{nr+1}{nr} \times$ into $\frac{b.\log v}{r} - \frac{1}{2v} - \frac{r}{12vv} : -Q$.

EXAM.

Suppose a life of 40 as before, $n = 48 = z$, $R = 1.04$, $r = .04$, $v = 2.96$, then $\frac{nr+1}{nr} = \frac{2.92}{1.92} = 1.5208$, and $\frac{z}{nr} = \frac{1}{r} = 25$; whence $Q = 25.755$, and

$\frac{nr+1}{nr} \times \frac{b.\log.v}{r} - \frac{1}{2v} - \frac{r}{12vv} : -Q = 40.996 - 25.755 = 15.24$, for the value of the annuity; which is something too great.

3. Or thus.

Let n be the complement of life, then sum up $n - 1$ terms in Table II, the sum of which divide by n; and the quotient will be the value of that life whose complement is n.

This is so easy it needs no example.

To prove the rule; suppose there are n persons, and that one dies every year, and each of them dies before the year be out. Then the first gets nothing; but the rest get the value of an annuity certain for 1, 2, 3, &c. years, to $n - 1$ years inclusive. The sum of all which is $n - 1$ terms of Table II. And since the number of men is n, therefore that sum is to be divided by n, for the mean value of a single life whose complement is n. And this exactly agrees with Cor. 2. before going.

SCHOL.

SCHOL.

This Prob. may be ſolved near enough, without introducing R^v into the computation, by finding the ſum of $\frac{n-1}{nR} + \frac{n-2}{nR^2} + \frac{n-3}{nR^3}$ &c. to n terms, and multiplying the laſt reſult by R^n or by $1 + vr$.

From this Prop. and its Cors. tables may be conſtrued, to ſhew the preſent worth of an annuity of 1*l.* for the life of any age; and ſuch a table you have farther on, for ſeveral rates of intereſt. And as the eaſieſt way of conſtructing ſuch tables, is by uſing p or P, that is, the preſent worth of 1*l.* or of the annuity certain of 1*l*; therefore I have alſo inſerted two tables for that purpoſe at the ſame rates of intereſt.

I muſt take notice that this Prob. may be ſolved, though not ſo exactly, by the *expectation* of life, thus. Find the value of the annuity certain, for the time $\frac{1}{2}n$, which in this example will be 15.25; but it will always give ſomething too much. The reaſon is, the former methods find the preſent worth of the annuity every year, and ſo gets the mean value. But the method by the expectation of life, only takes the preſent worth of the mean age, which is a different thing. But as the difference is not very great, it may ſerve in ſeveral caſes for an approximation.

If we ſuppoſe money to bear no intereſt; then in the preceding calculations, $R = 1$; and then we have nothing to do, but to ſum up an arithmetical progreſſion, which is $\frac{1}{2}n$, or half the complement of life; which ſhews that when money bears no intereſt, the value of a life coincides with the expectation of life, being equal to an annuity for $\frac{1}{2}n$ years, without intereſt. But this method applied to the foregoing example, would make the value

value of such a life 24*l.* which is far from the truth, being almost double. Therefore in using the expectation of life, we must allow interest.

PROP. XV. PROB.

The ages of two persons being given; to find the value of an annuity of 1l. for their joint lives; that is, till one of them dies.

1 *Way.*

Let the complements of life be m and n, n being the least. R = amount of 1*l.* for a year, r its interest. Then proceeding as before, the probability of each living a year being $\frac{m-1}{m}$, $\frac{n-1}{n}$; therefore the probability of both living 1, 2, 3, &c. years, will be $\frac{m-1}{m} \times \frac{n-1}{n}$, $\frac{m-2}{m} \times \frac{n-2}{n}$, $\frac{m-3}{m} \times \frac{n-3}{n}$ &c. whence the value of the joint lives will be $\frac{m-1}{m} \times \frac{n-1}{nR} + \frac{m-2}{m} \times \frac{n-2}{nR^2} + \frac{m-3}{m} \times \frac{n-3}{nR^3}$ &c. to n terms; for in n years one must necessarily die. That is,

$$\frac{1}{mn} \times \text{into } \frac{mn}{R} - 1. \frac{m+n}{R} + \frac{1}{R} + \frac{mn}{R^2} - 2. \frac{m+n}{R^2} + \frac{4}{R^2} + \frac{mn}{R^3} - 3. \frac{m+n}{R^3} + \frac{9}{R^3} \text{ &c.}$$

But the sum of the first column $= mn \times \frac{1-\frac{1}{R^n}}{r}$ $= mn \times \frac{1-p}{r}$ (by Prop. XXVI. proportion) and the

the second column $= \overline{m+n} \times \frac{pn}{r} + \frac{pR}{rr} - \frac{m+n}{rr}$ R (by Increments, Prop. XXX.)

And the third col. $= \frac{RR+R}{r^3} \longdash p \times : \frac{nn}{r} + \frac{2nR}{rr} + \frac{RR+R}{r^3}$: (ib.) and the whole divided by mn. Now the sum of all these is $\frac{1}{mn} \times$ into $\frac{mn}{r} - \frac{m+n}{rr} R + \frac{R+1}{r^3} R + \frac{m-n}{rr} pR - \frac{R+1}{r^3} pR$; where $p = \frac{1}{R^n}$ the present worth of 1*l.* due n years hence. Therefore the value of the annuity $= \frac{1}{mn} \times$ into $\frac{mn}{r} - \frac{m+n-\overline{m-n}.p}{rr} R + \frac{R+1}{r^3} \times R \times \overline{1-p}$. If the payments begin sooner than after a year, suppose sooner by the time v; then multiply the result by R^v, or by $1 + vr$. This is the most exact way of any.

Cor. 1. *If the lives are equal, the value of the joint lives will be* $= \frac{nr-2R}{nrr} + \frac{RR+R}{nnr^3} \times \overline{1-p}$, exactly.

2. *Otherwise thus.*

Let M, N, be the values of the two single lives, and put $q = \frac{1}{R^m}$, $s = 1-p$, $t = 1-q$, $\epsilon = p+1$. And assume $\frac{x}{\overline{M+1} \times \overline{N+1} - R \times MN}$ = value of the joint lives; to find x.

By

By the laſt Prop. $M = \frac{1}{r} \times \overline{1 - \frac{sR}{nr}}$, and $N = \frac{1}{r} \times \overline{1 - \frac{tR}{mr}}$, and $\overline{M+1} \times \overline{N+1} = \frac{RR}{rr} \times \overline{1 - \frac{s}{nr}} \times \overline{1 - \frac{t}{mr}} = \frac{RR}{rr} \times 1 - \frac{ms+nt}{mnr} + \frac{st}{mnrr}$.

And $RMN = \frac{R}{rr} \times 1 - \frac{ms+nt}{mnr} R + \frac{stRR}{mnrr}$,

therefore $\overline{M+1} \times \overline{N+1} - RMN = \frac{RR-R}{rr} + \frac{RR-R^3}{mnr^4} st = \frac{Rr}{rr} - \frac{rstRR}{mnr^3} = \frac{R}{r} - \frac{stRR}{mnr^3}$.

Therefore $\frac{x}{\overline{M+1} \times \overline{N+1} - RMN} = \frac{x}{\frac{R}{r} - \frac{stRR}{mnr^3}}$ = value of the annuity found the firſt way $= \frac{1}{r} - \frac{ms+nc}{mnrr} R + \frac{RR+R}{mnr^3} s = A$.

And $x = \frac{R}{rr} \times 1 - \frac{ms+nc}{mnr} R + \frac{RR+R}{mnrr} s - \frac{stR}{mnr} A$, from which ſubtract

$MNR = \frac{R}{rr} \times 1 - \frac{ms+nt}{mnr} R + \frac{stRR}{mnrr}$, and we have $x - MNR = \frac{R}{rr} \times - \frac{nc-nt}{mnr} R + \frac{RR+R-tRR}{mnrr} s - \frac{stR}{mnr} A$, therefore MNR falls ſhort of the true value x, by $\frac{RR}{rr} \times$ into $- \frac{p+q}{mr} + \frac{R+1-tR}{mnrr} s - \frac{st}{mnr} A = y$, whence

MNR

$\frac{MNR + y}{\overline{M+1} \times \overline{N+1} - MNR}$ = value of the two joint lives. But since y is always very small and affirmative, it may be set aside, and then the value of the joint lives will be $\frac{MNR}{\overline{M+1} \times \overline{N+1} - MNR}$, being a small matter too little, and is one of Mr. Moivre's rules.

It may here be observed, that since the correction y is irregular, the rule is not genuine; that is, not derived from true principles.

Cor. 2. *The value of the two joint lives is* = $\frac{MNR}{M+N-rMN}$, *nearly.*

For $\frac{MNR}{\overline{M+1} \times \overline{N+1} - RMN} = \frac{MNR}{MN+M+N+1-MN-rMN} = \frac{MNR}{M+N+1-rMN} = \frac{MNR}{M+N-rMN}$; for throwing out the 1 in the denominator, helps to correct it and bring it near the truth. Or as Mr. Moivre has it, $\frac{MN}{M+N-rMN}$, leaving out R. A very easy rule, but is something too little.

Cor. 3. *If the lives are equal, then the value of the joint lives will be* = $\frac{M}{2-rM}$, *or rather* $\frac{RM}{2-rM}$.

EXAM.

Let the ages be 40 and 50, interest at 5 *per* cent. Then $m = 48$, $n = 38$, $p = .1566$; and $\frac{1}{r} -$

$m +$

$\frac{m+n-\overline{m-n}.p}{mnrr}R = 36480 - \frac{90.3 - 1.6443}{.0025}$ $= 1017.72$, and $\frac{RR+R}{r^3} \times \overline{1-p} = 14523$, and $\frac{1017.72 + 14523}{mn} = 1824.$ $= 8.25$, for the value of the joint lives.

Or thus.

By Tab. III. $M = 1209$, and $N = 10.68$. Whence $MN = 129.12$, and $M + N = 22.77$. And $MNR = 135.57$, and $MNr = 6.45$; therefore $\frac{135.57}{22.77 - 6.45} = \frac{135.57}{16.32} = 8.30$, the value.

3. *Otherwiſe.*

The ſame may be alſo reſolved by the method of increments, another way, as in the laſt Prop.

The value of the joint lives was $\frac{m-1}{m} \times \frac{n-1}{nR} + \frac{m-2}{m} \times \frac{n-2}{nR^2} + \frac{m-3}{m} \times \frac{n-3}{nR^3}$ &c. $= \frac{m-1}{m} \times \frac{n-1}{n.1+r} + \frac{m-2}{m} \times \frac{n-2}{n.1+2r}$ &c. Then to find n terms or $z-1$ terms of this ſeries, we ſhall have the z^{th} term or $s = \frac{\overline{m-z} \times \overline{n-z}}{mn \times \overline{1+zr}}$, put $1 + zr = v$, then $z = \frac{v-1}{r}$, whence $s = \frac{\overline{rm+1-v} \times \overline{rn+1-v}}{rrmnv} = \frac{\overline{p-v} \times \overline{q-v}}{rrmnv}$, (by ſubſtitution) $= \frac{pq - \overline{p+q}.v + vv}{rrmnv} =$ (putting f

$=$

$= \overline{p+q}) \frac{1}{rrmn} \times$ into $v - f + \frac{pq}{v}$, and the integral $= \frac{1}{rrmn} \times$ into $\frac{\overset{v\,v}{-1}}{2r} - \frac{fv}{r} + pq \times \frac{b.\log.v}{r} - \frac{1}{2v} - \frac{r}{12vv}$, becauſe $v = r$. Or $s = \frac{1}{rrmn} \times$ into $\frac{v-r-2f}{2r} v + pq \times \frac{b.\,l.\,v}{r} - \frac{1}{2v}$ &c. but when s is $= 0$, $z - 1 = 0$, and $z = 1$, whence $v = 1 + r = R$; therefore the int. corrected is $s = \frac{1}{mnrr} \times$ into $\frac{v-r-2f.\,v + \overline{2f-1.R}}{2r} + pq \times : \frac{b.\,l.\,v - b.\,l.\,R}{r} + \frac{1}{2R} - \frac{1}{2v}$:

Thus in the former example, $m = 48$, $n = 38$, $r = .05$; $z - 1 = 38$, or $z = 39$, and $v = 2.95$, $rm = 2.4$, $rn = 1.9$, $f = 6.3$, $pq = 9.86$, and $rrmn = 4.56$, whence

$$\frac{v-r-2f.\,v + \overline{2f-1.R}}{2r} = -164.35$$

$$\text{and } pq \times \text{ \&c.} = 206.73$$

$$42.38$$

And $\frac{42.38}{4.56} = 9.10$ the value of the annuity, which is ſomething too great.

PROP. XVI. PROB.

The ages of three perſons being given, to find the value of an annuity of 1 *l. for their joint lives.*

1 *Way.*

Let the complements of their lives l, m, n; n being the leaſt. R the amount of 1*l.* for a year, r its

r its intereſt. Then by reaſoning as in the laſt Prop. The value of the joint lives will be $\frac{l-1}{l} \times \frac{m-1}{m} \times \frac{n-1}{nR} + \frac{l-2}{l} \times \frac{m-2}{m} \times \frac{n-2}{nR^2} + \frac{l-3}{l} \times \frac{m-3}{m} \times \frac{n-3}{nR^3}$ &c. to n terms: that is, (putting $S = lm + ln + mn$, $T = l + m + n$) the value $= \frac{1}{lmn} \times$ into

$$\begin{aligned} &\frac{lmn}{R} - \frac{1S}{R} + \frac{1T}{R} - \frac{1}{R} \\ &+ \frac{lmn}{R^2} - \frac{2S}{R^2} + \frac{4T}{R^2} - \frac{8}{R^2} \\ &+ \frac{lmn}{R^3} - \frac{3S}{R^3} + \frac{9T}{R^3} - \frac{27}{R^3} \\ &+ \frac{lmn}{R^4} - \frac{4S}{R^4} + \frac{16T}{R^4} - \frac{64}{R^4} \\ &\text{&c.} \end{aligned}$$

Then finding the ſum of each column (by the method of increments, Prob. XXX.) as in the laſt Prop. we ſhall have

1 col. $= lmn \times \frac{1-p}{r}$, where $p = \frac{1}{R^n}$.

2 col. $= Sp \times \frac{n}{r} + \frac{R}{rr} - \frac{SR}{rr}$.

3 col. $= T \times : \frac{R^2+R}{r^3} - p \times : \frac{nn}{r} + \frac{2nR}{rr} + \frac{R^2+R}{r^3}$.

4 col. $= -\frac{R^3+4R^2+R}{r^4} + p \times : \frac{n^3}{r} + \frac{3nnR}{rr} + \frac{3nR.R+1}{r^3} + \frac{R^3+4R^2+R}{r^4}$: and the whole ſum $=$

$= \frac{lmn}{r} + \frac{3n-2T}{rr} \times pnR - \frac{SR}{rr} \times \overline{1-p} +$
$\frac{R^2+R}{r^3} \times \overline{T-Tp+3np} - \frac{R^3+4R^2+R}{r^4} \times$
$\overline{1-p} = \frac{lmn}{r} + \frac{\overline{3n-2T} \times pnR - S \times \overline{1-p}}{rr} R$
$+ \frac{R^2+R}{r^3} \times \overline{T-Tp+3np} - \frac{R^3+4R^2+R}{r^4} \times$
$\overline{1-p}$. And dividing by lmn, the value of the joint lives will be $= \frac{1}{lmn} \times$ into $\frac{lmn}{r} +$
$\frac{\overline{3n-2T} \times pnR - RS \times \overline{1-p}}{rr} + \frac{RR+R}{r^3} \times$
$\overline{T-Tp+3np} - \frac{RR+4R+1}{r^4} \times R \times \overline{1-p}$.

Cor. 1. *If the lives are all equal; the value of the annuity upon three joint lives will be* $\frac{1}{n^3} \times$ *into* : $\frac{n^3}{r} -$
$\frac{3nnR}{rr} + \frac{3n \times \overline{RR+R}}{r^3} - \frac{R^2+4R+1}{r^4} R \times$
$\overline{1-p}$.

2. *Otherwiſe thus.*

Let M, N, Q, be the values of the ſingle lives, and Z a life equal to the joint lives M, N; then by the laſt Prop. Cor. 2. we ſhall find $Z = \frac{MN}{\overline{M+N-rMN}}$, and by the ſame, the value of the joint lives Z and Q, $\frac{MN}{M+N-rMN} \times Q$,

will be $\frac{ZQ}{Z+Q-rZQ} = \frac{\frac{MN}{M+N-rMN}+Q-rQ\times}{\frac{MN}{M+N-rMN}}$ = (multiplying by $M+N-rMN$)

$\frac{MNQ}{MN+Q\times M+N-rMN-rQ\times MN} = \frac{MNQ}{MN+MQ+NQ-2rMNQ}$; which is a rule of Mr. Moivre's, but is something too small, partly by leaving out R.

Cor. 2. *If all the lives are equal, the value of the three joint lives will be* $\frac{M}{3-2rM}$.

EXAM. I.

Suppose the three ages 45, 56, and 68; whose complements are $l = 43$, $m = 32$, $n = 20$. Interest at 4 per cent.

Here $lmn = 27520$, $p = .4564$, $1-p = .5436$, $R = 1.04$, $r = .04$, $S = 2876$, $T = 95$. Then

$$\frac{lmn}{r} = \frac{27520}{.04} = + 688000$$

$$+\frac{3n-2T\times pnR-SR\times\overline{1-p}}{rr} = - 1787000$$

$$+\frac{RR+R}{r^3}\times\overline{T-Tp+3np} = + 2619700$$

$$-\frac{R^2+4R+1}{r^4}\times\overline{1-p}.R = - 1378500$$

$$+ 3307700$$
$$- 3165500$$
$$142200$$

Then $\frac{142200}{27520} = 5.16$ the value of the annuity required.

3. *Or*

3. Or thus.

By Tab. III. M = 12.68, N = 10.47, Q = 7.33. Then

$$MNQ = 973.05,\quad MN = 132.74$$
$$2rMNQ = 77.84,\quad MQ = 92.94$$
$$NQ = 76.74.$$

$$MN + MQ + NQ = 302.43$$
$$-2rMNQ = -\ 77.84$$
$$224.59$$

Then $\frac{973.05}{224.59} = 4.33$ for the value of the three joint lives, which is too little; but if it be multiplied by $RR = 1.08$, the product is 4.67 more exact.

EXAM. II.

Suppose each life be 68; then $n = 20$, and $n^3 = 8000$. $R = 1.04$, $r = .04$. Then

$$\frac{n^3}{r} = + \ 200000$$
$$-\frac{3nnR}{rr} = -\ 780000$$
$$+\frac{3n \times \overline{R^2 + R}}{r^3} = + \ 1989000$$
$$-\frac{R^2+4R+1}{r^4} R \times \overline{1-p} = -\ 1378500$$
$$+\ 2189000$$
$$-\ 2158500$$
$$+\ 30500$$

Then $\frac{30500}{8000} = 3.81$, the value of the joint lives.

Or thus.

By Table III. M = 7·33 ; 7·33
.08

2r M = .5864
3.0000

2.4136

Then $\frac{7.33}{2.41} = 3.04$ the value of the joint lives, too little.

SCHOL.

Mr. Moivre has given another general rule by the expectation of life, which is this. Add 1 to each age, and then find the expectation of theſe joint lives, by Prop. 3d, 4th, or 5th. Find the complement of twice this number to 88, and take it as a ſingle life, which being found in Tab. III. will give the value required. At the ſame time he informs us, that he does not pretend that any of theſe rules are ſtrictly true, but only near approximations.

PROP. XVII. PROB.

To find the value of an annuity of 1l. for four, five, ſix, or any number of joint lives, of any ages.

1 *Way.*

This Prob. is to be reſolved by finding ſucceſſively the value of three lives, and then the value of four, and then of five, and ſo on, by the foregoing propoſitions. Thus, let A, B, C, D, E, &c. be any number of perſons, then

1. To find the value of the four lives ABCD; firſt find the value of the joint lives of the three ABC,

ABC, by the laſt Prop. and let Z be a ſingle life equal to theſe three. Then by Prop. XV. find the value of the joint lives of ZD, which will be the value of the four joint lives ABCD, by Prop. IX.

2. To find the value of the lives ABCDE. Having found the value of the four joint lives ABCD as before, put a ſingle life Z = theſe four; then by Prop. XV. find the value of the joint lives ZE, and that will be the value of the joint lives ABCDE, by Prop. IX.

3. Again, put Z = joint lives of ABCDE, and then find the joint lives of ZF, which will be the joint lives of ABCDEF. And thus you may go on as far as you will.

EXAM.

To find the value of the four joint lives whoſe ages are 31, 45, 56 and 68, at 4 per cent. The value of the three joint lives, whoſe ages are 45, 56 and 68, is 5.18 = Z, by the laſt Prop. Then by Table III. the value of a life of 31, is 14.81; then (by Cor. 2. Prop. XV.) $5.16 \times 14.81 = 76.42$, and $.04 \times 76.42 = 3.06$; and $5.16 + 14.81 - 3.06 = 16.91$; and $\frac{76.42}{16.91} = 4.52$, for the value of the four joint lives.

2. Or thus.

Seek the value of the life 5.16 (= Z) in Tab. III. againſt which, or the neareſt to it, is the age 75, whoſe complement is 13. Alſo the complement of the given age (31) is 57. Then by the two complements 13 (*n*), and 57 (*m*), the value of the joint lives 31 and 75 is eaſily found by Prop. XV. firſt way.

SCHOL.

To avoid a deal of trouble in finding the value of many joint lives, take the following approxima-

tion. Add all the ages together, which divide by the number of lives, the quotient is the mean age. By the help of this, find the value of the annuity, for ſo many equal joint lives, by the following propoſition.

PROP. XVIII. PROB.

To find the value of an annuity of 1*l. for ſeveral equal joint lives, whoſe number is m, and complement of life n.*

RULE.

The preſent value $= A - \frac{B}{n} + \frac{C}{nn} - \frac{D}{n^3} + \frac{E}{n^4} - \frac{F}{n^5} + \&c. + \frac{Z}{n^m} \times \overline{1 - p}$, continued to $m + 1$ terms. Where $p = \frac{1}{R^n}$, the preſent worth of 1*l.* due *n* years hence, and Z the laſt term, which is always to be ×d by $1 - p$. And the co-efficients are,

$rA = 1$, or $A = \frac{1}{r}$.

$rB = m \times \overline{A + 1}$, or $B = \frac{mR}{rr}$.

$rC = m.\frac{m-1}{2} \times \frac{R}{r} + \overline{m-1}.B$,

$rD = m.\frac{m-1}{2}.\frac{m-2}{3} \times \frac{R}{r} + \frac{m-1}{1}.\frac{m-2}{2}B + \overline{m-2}.C$.

$rE = m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{m-3}{4} \times \frac{R}{r} + \frac{m-1}{1}.$

$m -$

$\frac{m-2}{2} \cdot \frac{m-3}{3} B + \frac{m-2}{1} \cdot \frac{m-3}{2} C +$

$\overline{m-3}$. D. And so on, where the progression is evident.

This is proved by induction as follows; we have by the corollaries of Prop. 14, 15, 16, the value of

$$1 \text{ life} = \frac{1}{r} - \frac{R}{nrr} \times \overline{1-p}.$$

$$2 \text{ lives} = \frac{1}{r} - \frac{2R}{nrr} + \frac{RR+R}{n^2r^3} \times \overline{1-p}.$$

$$3 \text{ lives} = \frac{1}{r} - \frac{3R}{nrr} + \frac{3 \times \overline{RR+R}}{n^2r^3} - \frac{R^3+4R^2+R}{n^3r^4} \times \overline{1-p}.$$

&c. where the law of progression is evident. And the calculation of the quantities A, B, C, &c. is founded upon Prob. 30th of my *Increments*. For it will be found that $\frac{R}{rr}$, $\frac{RR+R}{r^3}$, $\frac{R^3+4RR+R}{r^4}$, &c. (the terms into which $1-p$ is multiplied) are the last terms in the series for summing up the laterals, squares, cubes, &c. as in Prob. 30, of the Increments. But it is not necessary to stay to reduce the coefficients to these forms, but take them as they lie in the rule, when you make use of it.

PROP. XIX. PROB.

To find the value of an annuity of 1l. upon the longest of two lives, whose ages are given.

1 *Way.*

Let m and n be the complements of their ages to 88. The probability of each living one year

will be $\frac{m-1}{m}$ and $\frac{n-1}{n}$, and the probability of each failing will be $1-\frac{m-1}{m}$, $1-\frac{n-1}{n}$, or $\frac{1}{m}$ and $\frac{1}{n}$; and the probability of both failing in a year will be $\frac{1}{m}\times\frac{1}{n}$ or $\frac{1}{mn}$, and therefore $1-\frac{1}{mn}$ is the probability of ſome being alive at the year's end.

Again, the probability of both failing the ſecond year will be $\frac{2}{m}\times\frac{2}{n}$ or $\frac{4}{mn}$, and $1-\frac{4}{mn}$ is the probability of ſome being alive at two year's end.

After the ſame manner the probability of ſome being living after 3, 4, &c. years will be $1-\frac{9}{mn}$, $1-\frac{16}{mn}$ &c.

Therefore the preſent values being found for 1, 2, 3, &c. years, the value of the annuity will be $\frac{1}{R}-\frac{1}{mnR}+\frac{1}{R^2}-\frac{4}{mnR^2}+\frac{1}{R^3}-\frac{9}{mnR^3}+\frac{1}{R^4}-\frac{16}{mnR^4}$ &c. to n terms at leaſt, in which time one muſt neceſſarily die. But ſince the other may live longer, the ſeries muſt be continued to more terms, put $v=\frac{m+n}{2}$, and the ſeries muſt be continued to v terms nearly. But this conſiſts of two ſeries $\frac{1}{R}+\frac{1}{R^2}+\frac{1}{R^3}$ &c. for v terms, $=1-\frac{\frac{1}{R^v}}{r}-\frac{1}{mn}$

$\frac{1}{mn} \times \frac{1}{R} + \frac{4}{R^2} + \frac{9}{R^3}$ &c. (v terms) $= \frac{p}{mn} \times \overline{\frac{vv}{r} + \frac{2Rv}{rr}} - \frac{1-p}{mn} \times \frac{R^2+R}{r^3}$, (Increments, Prob. 30.) putting $p = \frac{1}{R^v}$, to be found in Tab. I.

Therefore the value of the annuity $= \frac{1-p}{r} + \frac{p}{mn} \times \overline{\frac{vv}{r} + \frac{2Rv}{rr}} - \frac{1-p}{mn} \times \frac{R+1}{r^3} R$, a near approximation.

If the lives are equal, $m = n = v$.

2. *Or thus.*

Put M, N, for the values of the ſingle lives, j. $\overline{MN}$ for their joint lives; then by Cor. 1. Prop. VII. the value of the longeſt of the two lives is $= M + N - j. \overline{MN}$.

If the lives are equal, then the value of the longeſt life will be $2M - \frac{MR}{2 - rM}$.

EXAMP.

Let the ages be 45 and 56, then $m = 43$, $n = 32$, $v = 37\frac{1}{2}$. $R = 1.04$. $p = .230$, $1 - p = .770$.

Then $\frac{1-p}{r} =$	19.25
$\frac{p}{mn} \times \overline{\frac{vv}{r} + \frac{2Rv}{rr}} =$	14.02
	33.27
$- \frac{1-p}{mn} \times \frac{R+1}{r^3} R =$	— 18.55
The value required	14.72

Or

Or thus.

The value of the joint lives found by Prop. XV. is 8.28; then by Tab. III. M = 12.68
N = 10.47

23.15
— 8.28

The value required 14.87

PROP. XX. PROB.

To find the value of an annuity of 1*l. upon the longeſt of three lives, whoſe ages are given.*

1 *Way.*

Let l, m, n, be their complements to 88; then $\frac{l-1}{l}$, $\frac{m-1}{m}$, $\frac{n-1}{n}$ are the probabilities of living one year, and $1-\frac{l-1}{l}$, $1-\frac{m-1}{m}$, $1-\frac{n-1}{n}$, or $\frac{1}{l}$, $\frac{1}{m}$, $\frac{1}{n}$, are the probabilities of failing in a year, and $\frac{1}{lmn}$ = probability of all failing; and $1-\frac{1}{lmn}$ = probability of ſome living a year.

Likewiſe $\frac{2}{l}$, $\frac{2}{m}$, $\frac{2}{n}$ are the probabilities of failing the ſecond year, and $\frac{2}{l}\times\frac{2}{m}\times\frac{2}{n}$ or $\frac{8}{lmn}$ = probability of all failing the ſecond year; and $1-\frac{8}{lmn}$ = probability of ſome living two years.

Alſo

Alſo $1 - \frac{27}{lmn}$, $1 - \frac{64}{lmn}$ &c. = probabilities of ſome living 3, 4, &c. years. Then finding the preſent worth for 1, 2, 3, &c. years, and putting $v = \frac{l+m+n}{3}$; the value of the annuity will be $\frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3}$ &c. (for v terms) $= \frac{1-p}{r}$, where $p = \frac{1}{R^v}$.

$-\frac{1}{lmn} \times$ into $\frac{1}{R} + \frac{8}{R^2} + \frac{27}{R^3}$ &c. (v terms) $=$ (Prob. 30, Increments) $-\frac{1}{lmn} \times$ into $\frac{R^3+4R^2+R}{r^4}$

$-p \times : \frac{v^3}{r} + \frac{3vvR}{rr} + \frac{3v\,R \times \overline{R+1}}{r^3} + \frac{R^3+4R^2+R}{r^4}$: therefore the whole ſum, or value of the annuity is $= \frac{1-p}{r} + \frac{p}{lmn} \times : \frac{v^3}{r} + \frac{3vvR}{rr} + \frac{3vR}{r^3} \times \overline{R+1} + \frac{R^3+4R^2+R}{r^4}$:

$$- \frac{1}{lmn} \times \frac{R^3+4R^2+R}{r^4}.$$

2. *Or thus.*

This problem may alſo be ſolved pretty near the truth, by the expectation of life, thus;

Let l, m, n, be the complements of life as before, and x = years certain that the annuity is to continue. Then the probability that ſome one will be alive, and the reſt dead, at the end of x years, is $\frac{l-x}{lmn} xx + \frac{m-x}{lmn} xx + \frac{n-x}{lmn} xx$; and the probability

bability that they ſhall all be dead in x years, is $\frac{x^3}{lmn}$, which is equal to the other *per* queſt. Therefore $l-x+m-x+n-x=x$, and $4x=l+m+n$, and $x=\frac{l+m+n}{4}$. A near approximation, but ſomething too little.

By the ſame reaſoning if there were 4 lives, then $x=\frac{k+l+m+n}{5}$; and univerſally, if N be the number of lives, then $x=\frac{k+l+m+n\,\&c}{N+1}$; which will come the nearer to truth, as the number N is greater.

3. *Otherwiſe thus.*

By Prop. XIX. find a life Z equal to the longeſt of two of them A, B; and then (by the ſame Prop.) find the value of the longeſt of the two Z and C (a third life); and that will be the value of the annuity, by Cor. 1. Prop. VIII.

4. *Or thus.*

$A+B+C-j.\,AB-j.\,ZC=l.\,ABC$. That is, take the ſum of the three ſingle lives, from which ſubtract the ſum of the joint lives of AB, and of ZC, for the value of the longeſt of three lives.

For $l.\,AB=A+B-j.\,AB$, and $l.\,ABC$ or $l.\,ZC=Z+C-j.\,ZC=l.\,AB+C-j.\,ZC=A+B-j.\,AB+C-j.\,ZC$.

5. *Or thus.*

$A+B+C-j.\,AB-j.\,AC-j.\,BC+j.\,ABC$. That is, to the ſum of the three ſingle lives add the value of the three joint lives, from which

which ſubtract the ſum of the joint lives taken two and two. Which is Mr. Moivre's rule.

For by Cor. 4, Prop. IX. $j.\ ZC = j.\ AC + j.\ BC - j.\ ABC$, whence $A + B + C - j.\ AB - j.\ ZC = A + B + C - j.\ AB - j.\ AC - j.\ BC + j.\ ABC$.

EXAMP.

Let A, B, C, be three perſons whoſe ages are 45, 56, and 68; intereſt at 4 *per* cent. here $l = 43$, $m = 32$, $n = 20$, $v = \frac{l+m+n}{3} = 32$ nearly; whence $p = .289$, $1 - p = .711$; then

$$\frac{1-p}{r} = \frac{.711}{.04} = \quad + \quad 17.77$$

$$\frac{p}{lmn} \times \frac{v^3}{r} + \frac{3vvR}{rr} \text{ \&c.} = \quad 89.62$$

$$- \frac{R^3 + 4R^2 + R}{lmnr^4} = \quad - \quad 92.14$$

$$+ \quad 107.39$$

$$15.25$$ the value of the annuity.

Or thus.

Here $x = \frac{43 + 32 + 20}{4} = \frac{95}{4} = 24$ nearly. And by Tab. II. the preſent worth of an annuity certain, to continue 24 years (at 4 *per* cent.) is 15.24 for the value ſought.

Or thus.

By Tab. III. the value of the lives of 45 and 56, are 12.68 and 10.47; and (by Cor. 2. Prop. 15.) the value of the joint lives is 7.44; whence (Prop. XIX.) the value of the longeſt life of AB $= 23.15 - 7.44 = 15.71. = Z$; alſo the life C

(68) = 7.33; and the joint lives of ZC = 6.24; and then Z + C — $j.\ \overline{ZC}$ = 16.80 = $l.$ ZC = $l.$ ABC, that is, longeſt life of ABC = 16.80, the value of the annuity.

PROP. XXI. PROB.

To find the value of an annuity of 1l. for the longeſt of any number of lives, whoſe ages are given.

This performed by finding ſucceſſively the value for three lives (as above), then for 4 lives, then for five, and ſo on, by Prop. VIII. Thus, let A, B, C, D, E, F, &c. be ſeveral perſons, then,

1. Having found the value for the longeſt of three lives, as above, put Z = value for the three lives A, B, C; which reckon as a ſingle life. Then find the value of the longeſt life of ZD, by Prop. XIX. and that will be the value of the longeſt of the four lives ABCD.

2. Again, put Z = a ſingle life equal to the longeſt of the four ABCD, and find the value of the longeſt life of ZE as before, which will be the value of the longeſt of the five lives ABCDE. It is evident this proceſs may be carried on as far as you will with little trouble.

EXAMP.

Let the ages be 45, 56, 68, and 31, intereſt at 4 *per* cent. By the laſt Prop. the value of the longeſt of the three firſt lives is 15.25 = Z; and by Tab. III. the value of a life of 31, is 14.81 = E. Then $j.$ ZE = 10.74, and E + Z = 30.06, and 30.06 — 10.74 = 19.32 the value of the longeſt of the four lives.

PROP.

PROP. XXII. PROB.

To find the value of an annuity of 1*l. to continue as long as any two out of three perſons ſhall be alive.*

1 *Way.*

Let l, m, n, be the complements of life, and x the years certain for the continuance of the annuity. Then the probability of ſome two being alive and the third dead, (or perhaps all being alive) at the end of x years, will be $\frac{\overline{l-x}.\overline{m-x}.\overline{n-x}}{lmn} + \frac{\overline{l-x}.\overline{m-x}.x}{lmn} + \frac{\overline{l-x}.\overline{n-x}.x}{lmn} + \frac{\overline{m-x}.\overline{n-x}.x}{lmn}$ $= \frac{1}{2}$, by the nature of the queſtion, being an equal chance. Therefore

$$lmn - \overline{lm+ln+mn}.x + \overline{l+m+n}.xx - x^3 \Big\} + lm+ln+mn.x - 2l-2m-2n.xx + 3x^3 \Big\} = \tfrac{1}{2}lmn;$$

and reduced $2x^3 - \overline{l+m+n}.x^2 + \frac{1}{2}lmn = 0$, and $x^3 - \frac{l+m+n}{2}xx + \frac{1}{4}lmn = 0$.

And the root extracted will give x, a near approximation.

2. *Or thus.*

Let A, B, C, be three lives; the three joint lives takes away one life out of ABC. Another life is taken away by finding the reverſion for each joint life by Cor. 1. Prop. XI. thus,

$+ j.$ ABC $=$ 1 life extinct.
$+ j.$ AB $- j.$ ABC $=$ rev. $j.$ AB after the 1ſt life.
$+ j.$ AC $- j.$ ABC $=$ rev. $j.$ AC after the 1ſt life.
$+ j.$ BC $- j.$ ABC $=$ rev $j.$ BC after the 1ſt life.

Sum $j.\overline{AB} + j.\overline{AC} + j.\overline{BC} - 2j.\overline{ABC}$ $=$ total value of the annuity for two lives out of three.

EXAMP.

EXAMP.

Suppoſe three perſons A, B, C, whoſe ages are 45, 56, and 68; here $l = 43$, $m = 32$, $n = 20$. intereſt at 4 *per* cent. Then $x^3 - \frac{92}{2} xx + 6880 = 0$, and the root will be found $= 14.42$; and the preſent worth of an annuity to continue 14.42 years (by Tab. II.) is 10.79 the value of the two lives.

Or thus.

By Prop. XV. and XVI. we ſhall find,

$j.$ AB $=$	8.27	$\therefore j.$ ABC $= 5.16$,
$j.$ AC $=$	6.27	
$j.$ BC $=$	5.91	
	20.45	
$- 2j.$ ABC $=$	10.32	
	10.13	the value of the two lives.

PROP. XXIII. PROB.

To find the value of the reverſion of an annuity of 1*l. for one life after another, their ages being given.*

1 *Way.*

Let the two perſons be A, F; then by Cor. 1. Prop. X. the value of *l.* AF — life of A = reverſion of F after A. That is, from the preſent value of the longeſt liver, ſubtract the life of the poſſeſſor; and the remainder is the value of the reverſion due to F.

2. *Or thus.*

From the preſent value of the life of the expectant, ſubtract the preſent value of their joint lives; and

and the remainder is the value of the expectant's life after the poſſeſſor's.

For (by Prop. VII.) $l.\ AF + j.\ AF = A + F$, and $l.\ AF - A = F - j.\ AF =$ reverſion.

Cor. *Let* $A =$ *value of the poſſeſſor's life,*
$F =$ *value of the expectant's life,*
$r =$ *intereſt of* 1*l*; *then*

$\frac{1 - rA}{\frac{A}{F} + 1 - rA} \times F =$ *value of the expectation nearly, but too great.*

For (Cor. 2. Prop. XV.) $F - j.\ AF = F - \frac{AF}{A+F-rAF} = \frac{FF - rAFF}{A+F-rAF} = \frac{1 - rA}{A+F-rAF}$

$FF = \frac{1-rA}{\frac{A}{F}+1-rA} \times F.$

EXAMP.

Let the poſſeſſor's age be 56, and the expectant's 45; intereſt 4 *per* cent. then $A = 10.47$, $F = 12.68$; and $\frac{A}{F} = .83$, and $1 - rA = .58$ and $\frac{1-rA}{\frac{A}{F}+1-rA} \times F = 5.23$, the reverſion, but is too much; and will be more exactly done by finding the value of the joint lives by Prop. XV. firſt way, which will be 8.27; then

	F	$= 12.68$
	$j.\ AF$	$= -\ 8.27$
val. annuity		$= 4.41$

SCHOL. I.

If it was required to find the value of the annuity for the ſecond perſon and his heirs. From the value of the perpetuity ſubtract the value of the poſſeſſor's life.

SCHOL. II.

If it was required to find the value of the reverſion of an annuity, for the life of any perſon F, *after t years.*

From the value of F's life, (Tab. III.) ſubtract the value of the ſame life for *t* years certain (found by Cor. 1. Prop. XIII. annuities); the remainder is the reverſion.

PROP. XXIV. PROB.

To find the value of the reverſion of an annuity of 1*l. for one life after the longeſt of two, their ages being given.*

1 *Way.*

Let the perſons be A, B, F; F being the expectant. Then (by Cor. 3. Prop. X.) value of *l.* ABF — *l.* AB = reverſion of F after both the lives of AB. That is, if from the value of the longeſt life of all three, the value of the longeſt life of the two poſſeſſors be taken, the remainder is the value of the reverſion of one life after two.

2. *Or thus.*

The life of F — *j.* AF — *j.* BF + *j.* ABF = value. That is, to the life of the expectant, add the three joint lives; from which ſubtract the ſum of the joint lives of the expectant and each of the poſſeſſors; the remainder is the value of the reverſion.

For let Z = *l.* AB, then *l.* ABF = *l.* ZF = Z + F — *j.* ZF = Z + F — *j.* AF — *j.* BF + *j.* ABF. (by Cor. 4. Prop. IX); and therefore *l.* ABF — *l.* AB = F — *j.* AF — *j.* BF + *j.* ABF.

EXAMP.

EXAMP.

Let the ages of the poſſeſſors be 45 and 68, and the expectant 56; intereſt 4 *per* cent. Then by Prop. XIX. and XX.

	l. ABF =	15.24
—	*l.* AB =	13.74
value — reverſion		1.50

Or thus.

	F =	10.47
	j. ABF =	5.16
		15.61
	j. AF =	8.27
	j. BF =	5.90
		14.17
	reverſion	1.44

SCHOL.

If it was required to find the preſent value of an annuity for a life after two joint lives. By Cor. 3, Prop. XI. we ſhall have this rule, F — *j.* ABF = revers. F after *j.* AB. That is, from the value of the expectant's life, ſubtract the value of the three joint lives; the remainder is the reverſion.

PROP. XXV. PROB.

To find the value of the reverſion of an annuity of 1*l. for the longeſt liver of two perſons, after the life of one perſon; all their ages being given.*

Let the poſſeſſor be A, and expectants F, G. Then (by Cor. 2, Prop. X.) value of *l.* AFG —

life of A = reverſion of *l.* FG after the life of A. That is, from the value of the longeſt of the three lives, ſubtract the poſſeſſor's life; the remainder is the value of the reverſion.

EXAMP.

Let the poſſeſſor's age be 56, and the ages of the expectants 45 and 68. Intereſt 4 *per* cent. Then (by Prop. XX.) *l.* ABF = 15.24, and (by Tab. III.) life of A = 10.47,

l. ABF	=	15.24
life of A	=	10.47
reverſion		4.77

SCHOL.

If it was required to find the preſent value of an annuity, for two joint lives after the life of one perſon. Then (by Cor. 1. Prop. XI.) *j.* FG — *j.* AFG = value of the reverſion of *j.* FG after A. That is, from the value of the joint lives of the two expectants, ſubtract the value of the three joint lives, the remainder will be the reverſion.

PROP. XXVI. PROB.

To find how much the value of an annuity ought to be increaſed; when the payments are half-yearly or quarterly.

1 *Way.*

Suppoſe the year divided into m parts, and d = intereſt of 1*l.* for the m^{th} part of the year, and r = intereſt of 1*l.* for a year, t = number of years the annuity continues. Then,

$1+r$

$\frac{\overline{1+r}^{t}-1}{r}$ = value, for the yearly rents.

And $\frac{\overline{1+d}^{mt}-1}{md}$ = value, for the $\frac{1}{m}^{th}$ yearly rents.

Cor. $\frac{\overline{1+\frac{r}{2}}^{2t}-1}{r}$ = *value for half yearly payments.*

And $\frac{\overline{1+\frac{r}{4}}^{4t}-1}{r}$ = *value for quarterly payments.*

All this follows from Prob. 33, Algebra, augmenting the time and diminiſhing the intereſt and the rents, in the ſuppoſed ratio.

2. *Otherwiſe.*

Let V = value of the yearly payments, then $\frac{\overline{1+\frac{r}{m}}^{m}-1}{r}$ V = the value of the half yearly, or quarterly payments, according as $\frac{1}{m}$ denotes $\frac{1}{2}$ year or $\frac{1}{4}$ the year.

For r is the ſimple intereſt for a year, and $\overline{1+\frac{r}{m}}^{m}-1$ is the compound intereſt for all the m^{th} parts of it, through the year.

PROP. XXVII. PROB.

If A *enjoys an annuity of* 1*l. for his life, and at his death has the nomination of a ſucceſſor* B, *who is to enjoy the annuity for his life, both their ages being given; to find the value of the ſucceſſive lives.*

1 *Way.*

Let M be the value of A's life, and N the value of B's, both taken out of Tab. III. let V be the

preſent value of N, for A's life, then the preſent value of the ſucceſſive lives is $= M + V$. That is, find M in Table II, againſt which you have the year; find this year in Table I, againſt which is a number *(a)*, by which multiply N; then $M + aN$ = value of the ſucceſſive lives.

2. *Or thus.*

Let M, N, be the values of the two lives, as found in Table III. Alſo ſuppoſe the ſame values M, N, to be the values of an annuity certain for ſome numbers of years as *m*, *n*, (as in Table II.) Then (Algebra, Prob. 34, B. II.) $\frac{1-\frac{1}{R^m}}{r} = M$, and $1 - \frac{1}{R^m} = rM$, and $\frac{1}{R^m} = 1 - rM$; by the ſame reaſoning $\frac{1}{R^n} = 1 - rM$. Whence $\frac{1}{R^{m+n}} = 1 - rM - rN + rrMN$. Now let S = value of an annuity certain to continue $m + n$ years; then $\frac{1}{R^{m+n}} = 1 - rS = 1 - rM - rN + rrMN$; and reduced $S = M + N - rMN$, which is the thing required.

EXAM.

Let the ages be 40 and 60, then $M = 12.09$, $N = 8.83$, at 5 *per* cent. Then 12.09 found in Table II, gives 19 for the years; and 19 found in Table I, gives .3957, and $.3957 \times 8.83 = 3.49$; then $12.09 + 3.49 = 15.58$, the value required.

Or thus.

Here $MN = 106.7$, and $rMN = 5.34$. And $M + N = 20.92$, and $20.92 - 5.34 = 15.58$, as required.

PROP.

PROP. XXVIII. PROB.

If A *enjoys an annuity for his life, and at his death has the nomination of a ſucceſſor* B; *who alſo, at his death, has the nomination of a ſucceſſor* C, *who is to enjoy the annuity for his life. To find the value of the three ſucceſſive lives.*

1 *Way.*

Let M, N, Q, be the values of the three lives by Table III. Find N in Table II, againſt which you have the year. Find the year in Table I, againſt which is a number a, by which multiply Q; then $N+aQ$ is the value of theſe two, at the time of A's death.

In like manner find M in Table II, againſt which is the year; find the year in Table I, againſt which is a number b; by this multiply $N+aQ$; then $M+bN+abQ$ = value of the three ſucceſſive lives.

2. *Or thus.*

Suppoſe M, N, Q, to be the values of an annuity certain, each for ſome number of years, as m, n, q. And S the value of an annuity certain for $m+n+q$ years; then S will be the value required. Proceeding as in the laſt we ſhall have $\frac{1}{R^m} = 1-rM$, $\frac{1}{R^n} = 1-rN$, and $\frac{1}{R^q} = 1-rQ$.

Whence $\frac{1}{R^{m+n+q}}$ or $1-rS = \overline{1-rM} \times \overline{1-rN} \times \overline{1-rQ} = 1-rM-rN-rQ+rrMN+rrMQ+rrNQ-r^3MNQ$. Which being reduced we have $S = M+N+Q-r \times \overline{MN+MQ+NQ} + rrMNQ$.

EXAM.

Let the ages be 30, 40, and 60, at 5 *per* cent. then M = 13.19, N = 12.09, Q = 8.83. In Tab. II, 12.09 gives 19 for the year, and in Tab. I, 19 gives .3957 = a; and aQ = 3.49, and $N + aQ$ = 15.58.

Again, 13.19 in Table II, gives 22, and 22 in Table I, gives .3418 = b, then $M + b \times \overline{N + aQ}$ = 18.51, the value.

Or thus.

$M + N + Q$	=	34.11
$+ rr\,MNQ$	=	3.52
		37.63
$- r \times \overline{MN + MQ + NQ}$	=	19.13
The value required		18.50

Here follow the tables of the preſent value of 1 pound, and of 1 pound annuity certain, for any number of years under 90; and a Table of 1 pound annuity for any age. This laſt Table is conſtructed by Prop. XIV, or more eaſily by Cor. 2, by help of the firſt and ſecond Tables, which Tables may be proved true, or corrected where wrong, after this manner.

Let p = preſent worth of 1*l.* for any time t.
P = preſent worth of an annuity of 1*l.*
a = amount of 1*l.*
A = amount of 1*l.* annuity, all for t years.
r = rate of intereſt of 1*l.* for a year. All at compound intereſt.
$R = 1 + r$, the amount of 1*l.* for a year. Then,

1. ap

1. $ap = 1$.
2. $rA = a - 1$.
3. $\frac{1}{r}$ = perpetuity.
4. $A + a$ = amount of 1*l.* annuity for $t + 1$ years.
5. $P = \frac{A}{a} = pA = \frac{1-p}{r} = \frac{1}{r} - \frac{1}{rR^t}$.
6. Sum of n terms in Table I, = the n^{th} term in Table II, by the nature of annuities.
7. The ſum of $n - 1$ terms in Table II $= n \times$ num. in Table III, whoſe age is $88 - n$. See Prop. XIV.

Theſe equations ſhew the relation of the numbers in the two firſt Tables, and how one may be conſtructed from the other.

It appears by the Table Prop. I, that the numbers are irregular as far as 8 years of age. And therefore the numbers in Table III, under 8 are ſupplied by another rule, which is this. In the Table Prop. I, let a, b, c, &c. to f, be the numbers (or perſons living) againſt the ages A, $A + 1$, $A + 2$, &c. to $A + y$. And let V = preſent worth of an annuity for the age $A + y$. Then the preſent worth of an annuity for the age A is $= \frac{1}{a} \times$ into $\frac{b}{R} + \frac{c}{R^2} + \frac{d}{R^3}$ &c. to $\frac{f}{R^y} \times$ into $\overline{1 + V}$. Or to find them number by number, take only one term, then $\frac{b}{gR} \times \overline{1 + V}$ = preſent worth.

TABLE

TABLE I. *The preſent value of* 1 *pound, due any number of years hence. At* 3 *per cent. compound intereſt.*

years.	value.	years.	value.	years.	value.
1	.97087	31	.39999	61	.16479
2	.94259	32	.38834	62	.15999
3	.91514	33	.37702	63	.15533
4	.88848	34	.36604	64	.15080
5	.86261	35	.35538	65	.14641
6	.83748	36	.34503	66	.14215
7	.81309	37	.33498	67	.13801
8	.78941	38	.32522	68	.13399
9	.76642	39	.31575	69	.13008
10	.74409	40	.30656	70	.12630
11	.72242	41	.29763	71	.12262
12	.70138	42	.28896	72	.11905
13	.68095	43	.28054	73	.11558
14	.66112	44	.27237	74	.11221
15	.64186	45	.26444	75	.10894
16	.62317	46	.25674	76	.10577
17	.60502	47	.24926	77	.10269
18	.58739	48	.24200	78	.09970
19	.57028	49	.23495	79	.09679
20	.55367	50	.22811	80	.09398
21	.53755	51	.22146	81	.09124
22	.52189	52	.21501	82	.08858
23	.50669	53	.20875	83	.08600
24	.49193	54	.20267	84	.08350
25	.47760	55	.19677	85	.08106
26	.46369	56	.19103	86	.07870
27	.45019	57	.18547	87	.07641
28	.43708	58	.18007	88	.07418
29	.42434	59	.17482	89	.07203
30	.41199	60	.16973	90	.06993

TABLE I. *The present value of* 1 *pound, at* $3\frac{1}{2}$ *per cent, compound interest.*

age.	value.	age.	value.	age.	value.
1	.96618	31	.34423	61	.12264
2	.93351	32	.33259	62	.11849
3	.90194	33	.32134	63	.11449
4	.87144	34	.31047	64	.11061
5	.84197	35	.29998	65	.10587
6	.81350	36	.28983	66	.10326
7	.78599	37	.28003	67	.09977
8	.75941	38	.27056	68	.09639
9	.73373	39	.26141	69	.09313
10	.70892	40	.25257	70	.08998
11	.68494	41	.24403	71	.08694
12	.66178	42	.23578	72	.08400
13	.63940	43	.22780	73	.08116
14	.61778	44	.22010	74	.07842
15	.59689	45	.21265	75	.07576
16	.57670	46	.20547	76	.07320
17	.55720	47	.19852	77	.07073
18	.53836	48	.19180	78	.06833
19	.52015	49	.18532	79	.06602
20	.50256	50	.17905	80	.06379
21	.48557	51	.17300	81	.06163
22	.46915	52	.16715	82	.05955
23	.45328	53	.16149	83	.05754
24	.43796	54	.15603	84	.05559
25	.42315	55	.15076	85	.05371
26	.40884	56	.14566	86	.05189
27	.39501	57	.14073	87	.05014
28	.38165	58	.13597	88	.04844
29	.36875	59	.13138	89	.04681
30	.35628	60	.12693	90	.04522

TABLE I. *The preſent value of* 1 *pound, at* 4 *per cent. compound intereſt.*

years.	value.	years.	value.	years.	value.
1	.96154	31	.29646	61	.09140
2	.92455	32	.28506	62	.08789
3	.88899	33	.27409	63	.08451
4	.85480	34	.26355	64	.08126
5	.82193	35	.25341	65	.07813
6	.79031	36	.24367	66	.07513
7	.75992	37	.23430	67	.07224
8	.73069	38	.22528	68	.06946
9	.70259	39	.21662	69	.06679
10	.67556	40	.20829	70	.06422
11	.64958	41	.20028	71	.06175
12	.62460	42	.19257	72	.05937
13	.60057	43	.18517	73	.05709
14	.57747	44	.17804	74	.05489
15	.55526	45	.17120	75	.05278
16	.53391	46	.16461	76	.05075
17	.51337	47	.15828	77	.04880
18	.49363	48	.15219	78	.04692
19	.47464	49	.14634	79	.04512
20	.45638	50	.14071	80	.04338
21	.43883	51	.13530	81	.04171
22	.42195	52	.13010	82	.04011
23	.40572	53	.12509	83	.03857
24	.39012	54	.12028	84	.03708
25	.37512	55	.11565	85	.03566
26	.36069	56	.11121	86	.03429
27	.34682	57	.10693	87	.03297
28	.33348	58	.10282	88	.03170
29	.32065	59	.09886	89	.03048
30	.30832	60	.09506	90	.02931

TABLE I. *The preſent value of* 1 *pound, at* 4 ½ *per cent. compound intereſt.*

years.	value.	years.	value.	years.	value.
1	.95694	31	.25550	61	.06822
2	.91573	32	.24450	62	.06528
3	.87630	33	.23397	63	.06247
4	.83856	34	.22389	64	.05978
5	.80245	35	.21425	65	.05720
6	.76789	36	.20503	66	.05474
7	.73483	37	.19620	67	.05238
8	.70318	38	.18775	68	.05013
9	.67290	39	.17966	69	.04797
10	.64393	40	.17193	70	.04590
11	.61620	41	.16452	71	.04393
12	.58966	42	.15744	72	.04204
13	.56427	43	.15066	73	.04022
14	.53997	44	.14417	74	.03849
15	.51672	45	.13796	75	.03683
16	.49447	46	.13202	76	.03525
17	.47317	47	.12634	77	.03373
18	.45280	48	.12090	78	.03228
19	.43330	49	.11569	79	.03089
20	.41464	50	.11071	80	.02956
21	.39679	51	.10594	81	.02829
22	.37970	52	.10138	82	.02707
23	.36335	53	.09701	83	.02590
24	.34770	54	.09284	84	.02479
25	.33273	55	.08884	85	.02372
26	.31840	56	.08501	86	.02270
27	.30469	57	.08135	87	.02172
28	.29157	58	.07785	88	.02078
29	.27901	59	.07450	89	.01989
30	.26700	60	.07129	90	.01903

TABLE I. *The present worth of* 1 *pound, at* 5 *per cent. compound interest.*

years.	value.	years.	value.	years.	value.
1	.95238	31	.22036	61	.05098
2	.90703	32	.20986	62	.04856
3	.86384	33	.19987	63	.04624
4	.82270	34	.19035	64	.04404
5	.78352	35	.18129	65	.04194
6	.74621	36	.17266	66	.03995
7	.71068	37	.16443	67	.03805
8	.67684	38	.15660	68	.03623
9	.64461	39	.14915	69	.03451
10	.61391	40	.14204	70	.03286
11	.58468	41	.13528	71	.03130
12	.55684	42	.12884	72	.02981
13	.53032	43	.12270	73	.02839
14	.50507	44	.11686	74	.02704
15	.48102	45	.11130	75	.02575
16	.45811	46	.10600	76	.02452
17	.43630	47	.10095	77	.02336
18	.41552	48	.09614	78	.02224
19	.39573	49	.09156	79	.02118
20	.37689	50	.08720	80	.02018
21	.35894	51	.08305	81	.01921
22	.34185	52	.07909	82	.01830
23	.32557	53	.07533	83	.01743
24	.31007	54	.07174	84	.01660
25	.29530	55	.06832	85	.01581
26	.28124	56	.06507	86	.01505
27	.26785	57	.06197	87	.01434
28	.25509	58	.05902	88	.01366
29	.24294	59	.05621	89	.01300
30	.23138	60	.05353	90	.01239

TABLE II. *The preſent value of an annuity of* 1 *pound per annum, at* 3 *per cent, compound intereſt.*

years.	value.	years.	value.	years.	value.
1	0.97087	31	20.00043	61	27.84035
2	1.91347	32	20.38876	62	28.00034
3	2.82861	33	20.76579	63	28.15567
4	3.71710	34	21.13184	64	28.30648
5	4.57971	35	21.48722	65	28 45289
6	5.41719	36	21.83225	66	28.59504
7	6.23028	37	22.16723	67	28.73305
8	7.01969	38	22.49246	68	28.86704
9	7.78611	39	22.80821	69	28.99713
10	8.53020	40	23.11477	70	29.12342
11	9.25262	41	23.41240	71	29.24604
12	9.95400	42	23.70136	72	29.36509
13	10.63495	43	23.98190	73	29.48067
14	11.29607	44	24.25427	74	29.59288
15	11.93793	45	24.51871	75	29.70183
16	12.56110	46	24.77545	76	29.80760
17	13.16612	47	25.02471	77	29.91029
18	13.75351	48	25.26671	78	30.00999
19	14.32380	49	25.50166	79	30.10679
20	14.87747	50	25.72976	80	30.20076
21	15.41502	51	25.95123	81	30.29200
22	15.93692	52	26.16624	82	30.38059
23	16.44361	53	26.37499	83	30.46659
24	16.93554	54	26.57766	84	30.55009
25	17.41315	55	26.77443	85	30.63115
26	17.87684	56	26.96546	86	30.70986
27	18.32703	57	27.15093	87	30.78627
28	18.76411	58	27.33100	88	30.86045
29	19.18845	59	27.50583	89	30.93248
30	19.60044	60	27.67556	90	30.00241

TABLE

TABLE II. *The present value of an annuity of* 1 *pound per annum, at* 3 ½ *per cent. compound interest.*

years.	value.	years.	value.	years.	value.
1	0.96618	31	18.73627	61	25.06738
2	1.89969	32	19.06886	62	25.18587
3	2 80164	33	19.39021	63	25.30036
4	3.67308	34	19.70068	64	25.41098
5	4.51505	35	20.00066	65	25.51785
6	5.32855	36	20.29049	66	25.62112
7	6.11454	37	20.57052	67	25.72089
8	6.87395	38	20.84109	68	25.81727
9	7.60769	39	21.10250	69	25.91041
10	8.31660	40	21.35507	70	26.00040
11	9.00155	41	21.59910	71	26.08734
12	9.66333	42	21.83488	72	26.17134
13	10.30274	43	22.06269	73	26.25251
14	10.92052	44	22.28279	74	26.33092
15	11.51741	45	22.49545	75	26.40669
16	12.09412	46	22.70092	76	26.47989
17	12.65132	47	22.89944	77	26.55062
18	13.18968	48	23.09124	78	26.61896
19	13.70984	49	23.27656	79	26.68498
20	14.21240	50	23.45562	80	26.74878
21	14.69797	51	23.62861	81	26.81041
22	15.16712	52	23.79576	82	26.86996
23	15.62041	53	23.95726	83	26.92770
24	16.05837	54	24.11329	84	26.98309
25	16.48151	55	24.26405	85	27.03681
26	16.89035	56	24.40971	86	27.08870
27	17.28536	57	24.55045	87	27.13884
28	17.66702	58	24.68642	88	27.18729
29	18.03577	59	24.81780	89	27.23399
30	18.39204	60	24.94473	90	27.27932

TABLE II. *The present value of an annuity of* 1 *pound per annum, at* 4 *per cent. compound interest.*

years.	value.	years.	value.	years.	value.
1	0.96154	31	17.58849	61	22.71489
2	1.88609	32	17.87355	62	22.80278
3	2.77509	33	18.14764	63	22.88729
4	3.62989	34	18.41120	64	22.96855
5	4.45182	35	18.66461	65	23.04668
6	5.24214	36	18.90828	66	23.12181
7	6.00205	37	19.14258	67	23.19405
8	6.73274	38	19.36786	68	23.26351
9	7.43533	39	19.58448	69	23.33030
10	8.11089	40	19.79277	70	23.39451
11	8.76048	41	19.99305	71	23.45626
12	9.38507	42	20.18563	72	23.51564
13	9.98565	43	20.37079	73	23.57273
14	10.56312	44	20.54884	74	23.62762
15	11.11839	45	20.72004	75	23.68041
16	11.65229	46	20.88465	76	23.73116
17	12.16567	47	21.04293	77	23.77996
18	12.65930	48	21.19513	78	23.82689
19	13.13394	49	21.34147	79	23.87201
20	13.59032	50	21.48218	80	23.91539
21	14.02916	51	21.61748	81	23.95711
22	14.45111	52	21.74758	82	23.99722
23	14.85684	53	21.87267	83	24.03579
24	15.24696	54	21.99296	84	24.07287
25	15.62208	55	22.10861	85	24.10853
26	15.98277	56	22.21982	86	24.14282
27	16.32958	57	22.32675	87	24.17579
28	16.66306	58	22.42957	88	24.20749
29	16.98371	59	22.52843	89	24.23797
30	17.29203	60	22.62349	90	24.26728

TABLE II. *The present value of an annuity of* 1 *pound per annum, at* 4 ½ *per cent. compound interest.*

years.	value.	years.	value.	years.	value.
1	0.95694	31	16.54439	61	20.70625
2	1.87267	32	16.78889	62	20.77152
3	2.74896	33	17.02286	63	20.83399
4	3.58752	34	17.24676	64	20.89377
5	4.38998	35	17.46101	65	20.95098
6	5.15787	36	17.66604	66	21.00572
7	5.89270	37	17.86224	67	21.05811
8	6.59588	38	18.04999	68	21.10824
9	7.26879	39	18.22965	69	21.15621
10	7.91272	40	18.40158	70	21.20211
11	8.52892	41	18.56611	71	21.24604
12	9.11858	42	18.72355	72	21.28808
13	9.68285	43	18.87421	73	21.32830
14	10.22282	44	19.01838	74	21.36679
15	10.73954	45	19.15635	75	21.40363
16	11.23401	46	19.28837	76	21.43888
17	11.70719	47	19.41471	77	21.47261
18	12.15999	48	19.53561	78	21.50490
19	12.59329	49	19.65130	79	21.53578
20	13.00793	50	19.76201	80	21.56534
21	13.40472	51	19.86795	81	21.59363
22	13.78442	52	19.96933	82	21.62070
23	14.14777	53	20.06634	83	21.64660
24	14.49548	54	20.15918	84	21.67139
25	14.82821	55	20.24802	85	21.69511
26	15.14661	56	20.33303	86	21.71781
27	15.45130	57	20.41439	87	21.73953
28	15.74287	58	20.49223	88	21.76031
29	16.02189	59	20.56673	89	21.78020
30	16.28889	60	20.63802	90	21.79924

TABLE II. *The present worth of an annuity of* 1 *pound per annum, at* 5 *per cent. compound interest.*

years.	value.	years.	value.	years.	value.
1	0.95238	31	15.59281	61	18.98027
2	1.85941	32	15.80268	62	19.02883
3	2.72325	33	16.00255	63	19.07508
4	3.54595	34	16.19290	64	19.11912
5	4.32948	35	16.37419	65	19.16107
6	5.07569	36	16.54685	66	19.20102
7	5.78637	37	16.71129	67	19.23906
8	6.46321	38	16.86789	68	19.27530
9	7.10782	39	17.01704	69	19.30981
10	7.72173	40	17.15908	70	19.34268
11	8.30641	41	17.29437	71	19.37398
12	8.86325	42	17.42321	72	19.40379
13	9.39357	43	17.54591	73	19.43218
14	9.89864	44	17.66277	74	19.45922
15	10.37966	45	17.77407	75	19.48497
16	10.83777	46	17.88007	76	19.50949
17	11.27406	47	17.98101	77	19.53285
18	11.68959	48	18.07716	78	19.55510
19	12.08532	49	18.16872	79	19.57628
20	12.46221	50	18.25592	80	19.59646
21	12.82115	51	18.33898	81	19.61568
22	13.16300	52	18.41807	82	19.63398
23	13.48857	53	18.49340	83	19.65141
24	13.79864	54	18.56514	84	19.66801
25	14.09394	55	18.63347	85	19.68381
26	14.37518	56	18.69854	86	19.69887
27	14.64303	57	18.76052	87	19.71321
28	14.89813	58	18.81954	88	19.72687
29	15.14107	59	18.87575	89	19.73987
30	15.37245	60	18.92929	90	19.75226

TABLE III. *The present value of annuities upon a life of any age; at 3 per cent.*

age.	value.	age.	value.	age.	value.
0	14.53				
1	16.63	31	16.97	61	10.02
2	18.63	32	16.79	62	9.72
3	19.64	33	16.61	63	9.42
4	20.07	34	16.43	64	9.11
5	20.29	35	16.25	65	8.80
6	20.40	36	16.06	66	8.47
7	20.42	37	15.86	67	8.14
8	20.38	38	15.66	68	7.80
9	20.26	39	15.46	69	7.45
10	20.12	40	15.26	70	7.10
11	20.00	41	15.05	71	6.75
12	19.87	42	14.84	72	6.39
13	19.74	43	14.62	73	6.02
14	19.60	44	14.40	74	5.64
15	19.45	45	14.18	75	5.25
16	19.31	46	13.95	76	4.85
17	19.17	47	13.72	77	4.44
18	19.03	48	13.49	78	4.03
19	18.89	49	13.25	79	3.62
20	18.76	50	13.01	80	3.21
21	18.61	51	12.76	81	2.79
22	18.46	52	12.50	82	2.36
23	18.31	53	12 24	83	1.91
24	18.15	54	11.98	84	1.46
25	18.00	55	11.72	85	1.00
26	17.84	56	11.44	86	0.50
27	17.67	57	11.16	87	0.00
28	17.50	58	10 88		
29	17.32	59	10.60		
30	17.15	60	10.32		

TABLE

TABLE III. *The present value of annuities upon a life of any age; at* $3\frac{1}{2}$ *per cent.*

age.	value.	age.	value.	age.	value.
0	13.28				
1	15.21	31	15.84	61	9.64
2	17.02	32	15.69	62	9.36
3	17.95	33	15.53	63	9.07
4	18.35	34	15.37	64	8.78
5	18.56	35	15.20	65	8.49
6	18.67	36	15.03	66	8.18
7	18.70	37	14.86	67	7.86
8	18.68	38	14.69	68	7.54
9	18.58	39	14.52	69	7.22
10	18.48	40	14.34	70	6.90
11	18.37	41	14.16	71	6.57
12	18.26	42	13.97	72	6.23
13	18.15	43	13.78	73	5.87
14	18.04	44	13.59	74	5.51
15	17.93	45	13.40	75	5.14
16	17.82	46	13.20	76	4.76
17	17.70	47	12.99	77	4.37
18	17.58	48	12.78	78	3.97
19	17.46	49	12.57	79	3.57
20	17.34	50	12.35	80	3.16
21	17.22	51	12.13	81	2.74
22	17.10	52	11.90	82	2.31
23	16.98	53	11.67	83	1.87
24	16.85	54	11.44	84	1.42
25	16.73	55	11.20	85	0.98
26	16.58	56	10.96	86	0.49
27	16.43	57	10.71	87	0.00
28	16.28	58	10.45		
29	16.13	59	10.18		
30	15.98	60	9.91		

TABLE III. *The present value of annuities, upon a life of any age; at 4 per cent.*

age.	value.	age.	value.	age.	value.
0	12.23				
1	13.98	31	14.81	61	9.27
2	15.66	32	14.68	62	9.01
3	16.51	33	14.55	63	8.74
4	16.90	34	14.41	64	8.47
5	17.09	35	14.27	65	8.20
6	17.20	36	14.13	66	7.92
7	17.24	37	13.98	67	7.63
8	17.23	38	13.83	68	7.33
9	17.14	39	13.68	69	7.02
10	17.06	40	13.52	70	6.71
11	16.97	41	13.36	71	6.39
12	16.88	42	13.19	72	6.07
13	16.79	43	13.02	73	5.73
14	16.70	44	12.85	74	5.38
15	16.60	45	12.68	75	5.03
16	16.50	46	12.50	76	4.67
17	16.40	47	12.32	77	4.39
18	16.30	48	12.13	78	3.91
19	16.20	49	11.94	79	3.52
20	16.10	50	11.75	80	3.12
21	15.99	51	11.55	81	2.71
22	15.88	52	11.34	82	2.28
23	15.77	53	11.13	83	1.85
24	15.66	54	10 92	84	1.40
25	15.55	55	10.70	85	0.95
26	15.43	56	10.47	86	0.48
27	15.31	57	10.24	87	0.00
28	15.19	58	10.01		
29	15.07	59	9.77		
30	14.94	60	9.53		

TABLE

TABLE III. *The present value of annuities, upon a life of any age; at* $4\frac{1}{2}$ *per cent.*

age.	value.	age.	value.	age.	value.
0	11.31				
1	12.92	31	13.91	61	8.93
2	14.47	32	13.79	62	8.69
3	15.26	33	13.67	63	8.45
4	15.63	34	13.55	64	8.20
5	15.82	35	13.43	65	7.94
6	15.92	36	13.31	66	7.67
7	15.97	37	13.18	67	7.40
8	15.96	38	13.05	68	7.12
9	15.89	39	12.91	69	6.83
10	15.82	40	12.77	70	6.53
11	15.75	41	12.63	71	6.23
12	15.68	42	12.49	72	5.92
13	15.60	43	12.34	73	5.60
14	15.52	44	12.19	74	5.27
15	15.44	45	12.03	75	4.93
16	15.36	46	11.87	76	4.58
17	15.28	47	11.71	77	4.22
18	15.19	48	11.54	78	3.85
19	15.10	49	11.37	79	3.47
20	15.01	50	11.19	80	3.07
21	14.92	51	11.01	81	2.67
22	14.83	52	10.82	82	2.26
23	14.74	53	10.64	83	1.83
24	14.64	54	10.44	84	1.39
25	14.54	55	10.24	85	0.94
26	14.44	56	10.04	86	0.48
27	14.34	57	9.83	87	0.00
28	14.24	58	9.61		
29	14.13	59	9.39		
30	14.02	60	9.16		

TABLE III. *The present value of annuities, upon a life of any age; at 5 per cent.*

age	value.	age.	value.	age.	value.
0	10.52				
1	12.01	31	13.09	61	8.62
2	13.44	32	12.99	62	8.40
3	14.18	33	12.89	63	8.17
4	14.52	34	12.78	64	7.93
5	14.70	35	12.67	65	7.68
6	14.81	36	12.56	66	7.43
7	14.85	37	12.45	67	7.17
8	14.85	38	12.33	68	6.91
9	14.80	39	12.21	69	6.64
10	14.73	40	12.09	70	6.36
11	14.67	41	11.96	71	6.07
12	14.61	42	11.83	72	5.77
13	14.55	43	11.70	73	5.46
14	14.48	44	11.57	74	5.15
15	14.41	45	11.43	75	4.83
16	14.34	46	11.29	76	4.49
17	14.27	47	11.14	77	4.14
18	14.20	48	10.99	78	3.78
19	14.13	49	10.84	79	3.41
20	14.05	50	10.68	80	3.03
21	13.97	51	10.52	81	2.64
22	13.89	52	10.35	82	2.24
23	13.81	53	10.18	83	1.82
24	13.73	54	10.00	84	1.36
25	13.64	55	9.82	85	0.94
26	13.55	56	9.63	86	0.47
27	13.46	57	9.44	87	0.00
28	13.37	58	9.24		
29	13.28	59	9.04		
30	13.19	60	8.83		

ART.

ART. III.

The Inſtitution of Societies; or the Calculation of Annuities to be paid in any Society, inſtituted for the Benefit of old People, &c.

THE Calculation of Annuities for perſons entering into Societies, depends upon the Laws of Chance, and alſo upon the Rules of Annuities delivered in the laſt Article. But ſome queſtions of this ſort are ſo intricate, involve ſo many conditions, and require ſo many calculations, as to render them extremely difficult and laborious to perform. In ſuch caſes we muſt be content to make uſe of ſuch approximations as will ſhorten the work, and come near the truth. Every ſociety ought to be eſtabliſhed upon ſuch principles, that it may be able to continue for ever; and therefore all the conditions muſt be duly adjuſted to this end. And although, by the variableneſs and chances of things, the fund may ſometimes loſe; it will at other times gain as much, and ſo will never fail. Several Problems of this ſort here follow.

AXIOM I.

The preſent value of what is paid to the fund, muſt be at leaſt equal to the preſent value of what is received by the annuitants.

AXIOM II.

It is the ſame thing whether a number of people all enter at once, or ſome at one time and ſome at another.

PROB.

PROB. I.

A perſon whoſe age is a, puts a ſum of money s, into the hands of ſome ſociety, which ſum is to lie dead for y years, after which he is to receive an annuity for his life. To find that annuity, reckoning at 4 per cent.

Let $x =$ annuity, m the complement of a, and n the complement of $a+y$. Then ſuppoſe (Ax. 2.) there are m perſons, one will die every year; and in y years y of them will die, and receive no annuity. Now the money paid is $= ms$, and let the value of a life whoſe age is $a+y$ be $= v$, then $nv =$ money received by the remaining n people, this is, the ſum of all the annuities. Find p the preſent worth of nv due y years hence; then (Ax. 1.) $px = ms$, and $x = \frac{ms}{p}$, the annuity of each perſon.

EXAM.

Suppoſe a life of $42 = a$, and $y = 10$ years, $s = 50l.$ then $m = 46$, $n = 36$; and by Table III. a life of $a+y$ or $52 = 11.34 = v$, and $nv = 36 \times 11.34 = 408.38$; then the value of this 10 years ſooner is (by Table I.) $408.38 \times .6755 = 275.8 = p$, and $ms = 46 \times 50 = 2300$, and $x = \frac{2300}{275.8} = 8.33$, the annuity.

Therefore, if by this method the annuity for every age be computed, a table will be had, which will ſhew the annuity for any age.

Cor. 1. *If* $y = 0$, *and* $s = 11.34$; *then* $x = 1$, *for the annuity.*

Cor.

Cor. 2. *If x be given, and the fine s be required, then* $s = \frac{px}{m}$.

SCHOL.

If no intereſt be reckoned, then the money paid will be 2300 as before. But the money received will be $1+2+3+4$ &c. to $35 = \frac{35\times 36}{2} = 630$; whence $630x = 2300$, and $x = \frac{2300}{630} = 3.65$. Therefore the greateſt part of the annuity ariſes from the intereſt.

PROB. II.

A perſon whoſe age is a, pays no fine, but pays b pounds a year to ſome ſociety for y years. To find what annuity he muſt have afterwards for his life, at 4 per cent. (London Union Society.)

Let x = annuity, m = compl. of a, n = comp. of $a+y$. Then ſuppoſe m perſons to enter, y of them will die in y years.

1. Then the money paid by the dead men y, is to be found by Cor. 2. Prop. XIV. Annuities, putting y for n. Or (which is the ſame thing,) ſeek the age $\overline{88-y}$ in Table III. againſt which is v, the value of that life; this multiplied by y, that is vy = money paid by the y dead men, at 1*l.* annuity. And bvy = whole that is paid by theſe dead men.

Then to find what is paid by the remaining x perſons; here each of theſe perſons pays an annuity certain for y years. In Table II. againſt y find p the preſent worth, then bpx = ſum paid by the x perſons. Whence $bvy + bpx$ = whole ſum paid by all.

2. The

2. The money received is the annuity x, for 1, 2, 3, 4, &c. to $n-1$ years $= nq$, q being the present value of a life of the age $a+y$. Then $qnx =$ money received after y years. Find t in Table I. the present worth of 1*l.* due y years hence. Then $qnxt =$ present worth of all the money received; whence (Ax. 1.) $qnxt = bvy + bpn$, and $x = \frac{bvy+bpn}{qnt}$ the annuity sought.

EXAM.

Let the age be $42 = a$, $b = 1$, $y = 10$; then $m = 46$, $n = 36$.

The age $88 - y$, or 78 in Table III. gives $v = 3.91$, then $vy = 39.10$, and $bvy = 39.10$.

Again, Table II. against 10 is $8.11 = p$, and then $bpn = 291.96$ and $bvy + bpn = 331.06$, all the money paid.

Then $a + y = 52$. and against 52 (Table III.) is $11.34 = q$, and $t = .6755$ by Table I. whence $qnt = 275.7$, and $x = \frac{331.06}{275.7} = 1.20$, the annuity received.

SCHOL.

By the help of these two problems, that case may be solved, when both a fine and an annuity is to be paid into the fund; and the annuity required, for the surviving members. As in the last example, let $f =$ fine or præmium that each pays, then $46f$ or mf is to be added to the money paid, and we shall have $x = \frac{mf+bvy+bpn}{qnt}$. And if the annuity received (x) be given, the annuity (b) to be paid, may be found. (York Universal Society.)

PROB.

PROB. III.

A perſon of the age a, enters into a ſociety (inſtituted for young people,) and pays b pounds a year for y years, and then lies dead for z years. To find what annuity, he ſhall have afterwards for his life, at 4 per cent.

Let x annuity to be received, $m =$ comp. a, $n =$ comp. $a + y$, $w =$ comp. $a + y + z$. Then ſuppoſe m perſons in the beginning; then y of them will die in y years, and $y + z$ in $y + z$ years.

1. The money paid by the dead men is found thus, ſeek the number $88 - y$ in Table III. againſt which is v, then $bvy =$ waſte money paid by the y dead people.

For the other n perſons, againſt y in Table II. find p; then $bpn =$ money paid by theſe n perſons, and the whole is $= bvy + bpn$.

2. The money received is the annuity x for 1, 2, 3, &c. to $w - 1$ years; and this is $= wq$, where q is the value of a life whoſe age is $a + y + z$, found in Table III. Therefore $qwx =$ money received after $y + w$ years. In Table I. find t the preſent worth of 1*l.* due $y + z$ years hence: then $qwxt =$ preſent worth of all the money received for the annuitants. Whence $qwxt = bvy + bpn$, and $x = \frac{bvy + bpn}{qwt}$, the annuity for each perſon.

EXAMPLE.

Suppoſe the age be 40 $= a$, $b = 5$, $y = 7$, $z = 6$, and $m = 48$, $n = 41$, $w = 35$, $88 - y = 81$, $a + y + z = 53$, $y + z = 13$.

1. Againſt 81 (Table III.) is $2.71 = v$, then $bvy = 94.85$ the dead money.

Againſt

Againſt 7 (Table II.) find 6.002 $= p$, then $bpn = 1230.41$; and the whole ſum $bvy + bpn = 1325.26$.

2. Againſt 53 (Table III.) is 11.13 $= q$, and againſt 13 (Table I.) is .6005 $= t$. Then $qwt = 233.9$; whence $x = \frac{1325.2}{233.9} = 5.67$, the annuity.

PROB IV.

A perſon whoſe age is a, enters into ſociety, and pays f for a fine, and the annuity certain of b pounds a year, for y years. And afterwards a perſon of the age c (nominated at firſt,) is to receive an annuity for his own life. But if the nominee dies within the term, the payment ceaſes. To find the annuity, at 4 per cent.
(York Union Society.)

1. Let $x =$ annuity, $m =$ comp. a, $n =$ comp. $c + y$. Suppoſe m contributors at firſt, then $mf =$ ſum of all the fines: and then y nominees will die in y years; and then the y contributors will pay the annuity b for 1, 2, 3 — to $y - 1$ years $= byA$, where A $=$ number in Table III. againſt the age $88 - y$.

The other n contributors will pay nvb, where v is the number in Table II. againſt y. Therefore $mf + byA + bnv =$ whole money paid.

2. The n nominees remaining will receive the annuity x for 1, 2, 3—to $n-1$ years $= tpnx$, where $t =$ number againſt y in Table I. And p is the number againſt $c + y$ in Table III. Then $ptnx = mf + byA + bnv$, and $x = \frac{mf + byA + bnv}{ptn}$.

EXAM.

EXAM.

Let $a=40$, $f=10$, $b=5$, $y=17$, $c=30$, $m=48$, $n=41$. Then $mf=480$. Againſt 71 (88—y) in Table III. is 6.39=A, then byA=543.15.

Againſt 17 (Table II.) is 12.165=v, then bnv =2493.82; the ſum is 3516.97.

Againſt 47 ($c+y$) in Table III. is 12.32 = p. And againſt 17 in Table I. is .5134 = t. Whence $ptn = 259.33$, and $x = \frac{3516.97}{259.33} = 13.56$.

PROB. V.

A man and his wife enter into a ſociety, on theſe terms; that every man ſhall pay a præmium or admiſſion fine f, and b pounds a year, till one of them be dead. After the huſband's death, the wife (if living) is to receive an annuity for her life. To find this annuity; ſuppoſing money at 4 per cent.

(Laudable Society.)

Let a be the man's age, c the woman's, x = annuity.

This Prob. is ſolved by finding the value of the reverſion of an annuity for one life after another, by Prop. XXIII. Annuities. Therefore (by Prop. XV. ib.) find G the joint lives of the two perſons. And by Table III. find W the value of the wife's life; then W — G = value of the reverſion, at 1*l. per* annum.

It is evident, that $f+b$G (the value of the joint lives) is all the money paid by the huſband; and the reverſion $\overline{W-G} \times x$ is all the money received by the wife. Therefore $\overline{W-G} \times x = b$G $+ f$, and $x = \frac{f+Gb}{\overline{W-G}}$, the annuity.

If

If both husband and wife are to have the chance, then it is evident, the annuity will be but half so much.

EXAM.

Let $a=40$, $c=30$, $f=10$, $b=5$, then the value of the joint lives is $10.28=G$, and $W=14.94$; then $x=\frac{61.40}{4\ 66}=13.18$, the annuity.

If a and $c=40$, f and $b=1$. Then $W=13.52$, $G=9.63$, (by Cor. 3. Prop. 15. annuities,) whence $x=\frac{10.63}{3.89}=2.73$: but finding $G=10.12$ (by Rule 1. Prop. 15.) then x will come out 3.27, more exact.

If women live longer than men, or there be more widows than widowers, as some assert; suppose as 3 to 2; then instead of $W-G$, take $\overline{W-G}\times 1+\frac{3-2}{3+2}$ or $\overline{W-G}\times 1\frac{1}{5}$.

SCHOL.

If it be required to find how many contributors and annuitants there are in any year. It may be found thus, when the lives are equal; as suppose they are both 40. Here 48 is the comp. life; therefore suppose 48 persons of either sort, then one will die constantly in a year. Let $n=48$, and $d=$ men or women dead in d years, and l (or $n-d$) $=$ those living.

Since the living or dead belong promiscuously to the whole number, therefore find by proportion (by Prop. VII. *Laws of Chance*) how many living men belong to the living women. That part will be $\frac{l}{n}$; that is, out of 48, $\frac{ll}{n}$ men have wives, and therefore that is the number of contributors at d years.

Again,

Again, find how many living women there are belonging to the dead men; and $\frac{d}{n} \times l$ is the number, and this is the number of widows. Whence after d years, there are $\frac{ll}{n}$ contributors, and $\frac{dl}{n}$ annuitants, let d be what it will: and conſequently the number of contributors will then be to the number of widows, as l to d.

And from hence, by help of Table I. one may compute what the widow's annuity will be, if any one will be at the pains to do it, after the following manner. Let the annuity paid by each contributor be 1*l*. Then make a column of all years to 48, againſt which put down for each year $\frac{ll}{n} \times$ by the correſpondent number of Table I. The ſum of all theſe 48 numbers added to the præmium, will be the preſent worth of all the money paid, during the whole term.

Again, put down againſt theſe, in another column, for each year $\frac{ld}{n} \times$ by the correſpondent number of Table I. (as in the former column,) and the ſum of all theſe numbers $\times$ d by x, will be the preſent worth of all the money received by the widows. And therefore the common annuity x will be known by diviſion, and this is at 1*l*. annual contribution.

Purſuing this method for the age of 40 for man and wife, and 1*l*. fine and annuity; I found the widow's annuity 3*l*.27; agreeable to what was ſhewn before.

And hence it follows, that in this ſcheme, the number of contributors $\left(\frac{ll}{n}\right)$ is greateſt at firſt, and continually decreaſes to the laſt.

And the number of widows $\left(\frac{dl}{n}\right)$, is greateſt when $d=l$, that is, when d or $l=\frac{1}{2}n$, and then the widows and contributors are equal; and each number is $=\frac{1}{4}n$. Afterwards they continually decreaſe to 0.

PROB. VI.

A man and his wife of the ſame age (between 21 and 45) enter into ſociety; and pay 2l. admiſſion fine, and 1l. a year for his life. And afterwards the widow is to enjoy an annuity for her life; which will be more or leſs, according as her huſband has contributed a longer or a ſhorter time; that is, for 1, 7, or 15 years. To find the annuity to be paid in each of theſe intervals, at 4 per cent. (London Annuity Society.)

The age of the perſons muſt be given, or elſe the Prob. is unlimited. Suppoſe the age 40, the complement 48.

Let $a=$ age, $y=$ time of continuance, $m=$ comp. a, $n=$ comp. $a+y$, $f=$ fine, $x=$ annuity, $v=$ value of an annuity certain for y years, Table II. $p=$ value of the life $a+y$, Table III. $t=$ number againſt y, Table I.

Suppoſe there be m contributors at firſt; at the end of y years, there will only be n contributors; therefore at a mean, through the time y, there will be $\frac{m+n}{2}$ contributors. And at the end of y years, there will be n widows; but ſuppoſing only one widow, then we muſt reckon $\frac{m+n}{2n}$ contributors, or which is the ſame thing, one contributor and $\frac{m+n}{2n}$ $\times$ an-

x annuity paid, put $\frac{m+n}{2n} = s$. Then will $x = \frac{f+sv}{pt}$.

1. Let the time be 4 years, the middle between 1 and 7; then $m=48$, $y=4$, $n=44$, $a+y=44$, $f=2$, $v=3.63$, $p=12.85$, $t=.855$, $s=1\frac{1}{22}$. Then $x=\frac{5.795}{10.98}=.52$, the widow's annuity from 1 to 7 years.

2. The middle between 7 and 15 is $11=y$, then $a+y=51$, $n=37$, $v=8.76$, $p=11.55$, $s=1.16$, $t=.649$; then $x=\frac{12.15}{7.50}=1.60$, the annuity from 7 to 15.

3. The middle between 15 and 48 is $32=y$, and $a+y=72$, $n=16$, $v=17.87$, $p=6.07$, $t=.285$, $s=2$. Then $x=\frac{37.74}{1.73}=21.81$, the annuity from 15 to the end.

This Prob. may also be solved by Prob. IV. But this scheme is very imperfect. For the annuity should be computed for every year, or at least for every 5 years; and that for all ages between 21 and 45.

PROB. VII.

A person enters into society, and pays a fine f, and b pounds a year for y years; after which he is to receive an annuity for life. But if he dies before y years be out, the payment ceases. To find this annuity for life; at 4 per cent. (Provident Society.)

This Prob. is resolved like the rest. Let $a=$ the age, m its complement. Then if there be m people at first, y of them will die in y years. And the money paid by these, excluding the fine, is $=yA$,

where A is the value of a life whoſe complement is y. This is at $1l.$ a year.

If $n =$ comp. $a + y$; then theſe n contributors (in y years) pay nv, where v is the preſent value of $1l.$ annuity for y years.

After y years, there are alſo n annuitants. And they in all receive np, where p is the value of a life whoſe comp. is n. But the value of this, y years ſooner, is tnp, where t is the preſent worth of $1l.$ due y years hence.

But to reduce all to one perſon's annuity $= x$; divide by n, and then the money paid is $= \frac{m}{n} f + \frac{by\mathrm{A}}{n} + bv$; and the money received $= tpx$.

Therefore put $m =$ comp. of a,
$n =$ comp. $a + y$.
$\mathrm{A} =$ number in Table III. againſt $88 - y$.
$v =$ number in Table II. againſt y.
$p =$ number in Table III. againſt $a + y$.
$t =$ number in Table I. againſt y.

Then the annuity $x = \dfrac{\frac{m}{n} f + \frac{y}{n}\mathrm{A} + v \times b}{pt}$.

EXAM.

Suppoſe the age $40 = a$, and $y = 10$, $f = 0$, $b = 12.8$; then $m = 48$, $n = 38$, $\mathrm{A} = 3.92$, $v = 8.11$, $p = 11.75$, $t = .6755$; then $x = \dfrac{117.017}{7.93} = 14.74$ the annuity.

If $f = 31\frac{1}{2}$, $b = 8\frac{2}{3}$, then $x = 14.69$, the annuity.

Cor. *If the conditions be to pay f fine, and b pounds a year, for half the complement of life (to 88); then* $y = \frac{1}{2} m$, *and* $\mathrm{A} = p$. Whence $x = \dfrac{2f + \mathrm{A} + v.b}{\mathrm{A}t}$.

And this is a very rational ſcheme to proceed on.

PROB.

PROB. VIII.

A man whoſe age is a, enters into ſociety, and pays 1*l. a year for life. What ſum of money ſhall his heir &c. receive at his death; reckoning* 4 *per cent.*
(Amicable Society.)

Let $n = 88 - a$, ſeek a in Table III. and put $s =$ its value. And ſuppoſe there are n perſons at firſt, then 1 will die every year; and the amount of 1*l.* preſent money for 1, 2, 3, &c. to n years, will be $R + R^2 + R^3$ &c. to R^n, whoſe ſum is $\frac{R^n - 1}{r} R$ for n perſons, and the mean of all theſe ſums is $\frac{R^n - 1}{nr} R$, or putting $M =$ amount of 1*l.* for n years, by Table I. then the mean $= \frac{M-1}{nr} R$ the ſum that his heir, &c. ſhall have at his death, for 1*l.* paid at firſt. And conſequently $\frac{M-1}{nr} RS$ is the ſum that his heir receives at his death for paying 1*l.* annuity for his life, or S at firſt.

EXAM.

Suppoſe a life of 32, its value $S = 14.68$, at 1*l.* a year for life; then $n = 56$, and $M = 8.992$, then $\frac{M-1}{nr} R = 3.71$; and $3.71 \times 14.68 = 54.46 =$ ſum paid at his death, which is too much.

Or thus.

Suppoſe there be n perſons, and each pays 1*l.* at his death; then the preſent worth of 1*l.* due 1, 2,

3, &c. years hence is $= \frac{1}{R} + \frac{1}{R^2} + \frac{1}{R^3}$ &c. to $\frac{1}{R^n}$ whose sum $= \frac{1 - \frac{1}{R^n}}{r} = \frac{1-p}{r}$; and the mean present worth of 1*l.* for each person, is $= \frac{1-p}{nr}$, where $p =$ present worth of 1*l.* due n years hence. But $\frac{1-p}{nr} : 1l :: 1l : \frac{nr}{1-p}$ = money paid at his death, supposing 1*l.* paid at first. Or $\frac{nr}{1-p}$ S = money paid at his death, supposing S (or 1*l.* annuity) paid at first.

In a life of 32, $\frac{nr}{1-p} = 2l.52$ the money paid at last, for 1*l.* paid at first.

PROB. IX.

The same things supposed as in Prob. V. *and that all the members are of the same age at first; and also that the same number of members is constantly kept up, by admitting new ones, when the old ones die. To find the number of contributors, and number of annuitants in any one year.*

In the Schol. Prob. V. I shewed how to find the number of contributors and annuitants in any year, when no new members were admitted; in which case, one will die every year out of n (or 48), and therefore all will be dead in 48 years, the complement of life. But here as there are more members coming in, they will die faster and faster, till at last, or at the end of n (or 48) years, there will be people of all ages from 40, the year of admission, to the end of life; at which time, two will continually die every year, which I thus prove.

1. Sup-

1. Suppoſe there are n (or 48) couple, of the age of 40, 47 of the age of 41, 46 of the age of 42, 45 of the age of 43; and ſo on to 1. Then it is plain, by the laws of mortality, that one will die out of every parcel, every year; and therefore 48 will die in a year, out of the whole number; that is, out of $1+2+3+4$ &c. to 48, or 24×48; and conſequently 1 out of 24 will die in a year, or 2 out of 48. And therefore one woman will die every half year. And always at the death of a woman a new couple enters.

2. To find the number of contributors yearly. By Cor. Prop. IV. annuities, the expectation of two equal joint lives is $\frac{1}{3}n$; and this muſt be ſuppoſed to be the life of a contributor. For in a great number, if ſome be longer, others will be ſhorter; and after $\frac{1}{3}n$ years, they will go off as faſt as they come on; and after that will admit of no increaſe or diminution; therefore there will be then conſtantly $2\times\frac{1}{3}n$, or $\frac{2}{3}n$ contributors.

3. For the number of widows. By Cor. 1, Prop. VI. annuities, the expectation of the longeſt of two lives is $\frac{2}{3}n$; therefore $\frac{2}{3}n$ is the life of the longeſt liver, be it man or woman, and therefore $\frac{1}{3}n$ is the length of the life of the ſurvivor, after the death of the other; conſequently in theſe $\frac{1}{3}n$ years $\frac{2}{3}n$ ſurvivors will come in, as before of widows, and conſtantly be the ſame afterwards. But as half of this number is the number of ſurviving men, the other half or $\frac{1}{3}n$ will be the number of ſurviving women; that is, there will be conſtantly $\frac{1}{3}n$ widows or annuitants. All this will be more clear, by conſidering, that there are conſtantly n women, wives and widows; and ſo many wives ſo many contributors; and $\frac{2}{3}n$ contributors $+\frac{1}{3}n$ annuitants $=n$, the whole number.

4. To make an eſtimate for the intermediate years. Let d denote any year from the beginning;

then by a like computation I find, that after d years, 1 out of $n - \frac{1}{2}d$ perſons, will die in a year, of the ſupplemental people; or that $\frac{d}{n-\frac{1}{2}d}$ perſons will die in a year.

In the ſeries A below, one perſon dies every year, which is to be ſupplied by a new one coming in. But inſtead of one, let 48 come in, (that 1 may die in a year), as is denoted by the ſeries B, C, D, &c. out of each of which alſo 1 dies in a year.

A = 48, 47, 46, 45, 44, 43, 42, &c.
B = 48, 47, 46, 45, 44,
C = 48, 47, 46, 45,
D = 48, 47, 46,

&c. Now proceeding as in Schol. Prop. V. aſſume any number for d, the year. The number for d in the ſeries A is $\frac{ll}{n}$. For $d-1$, in the ſeries B, is $\frac{\overline{l+1}^2}{n}$. For $d-2$, in ſeries C is $\frac{\overline{l+2}^2}{n}$. For $d-3$, in ſeries D, is $\frac{\overline{l+3}^2}{n}$ &c. for $d-1$ terms beginning at B. The ſum of all which (divided by n) is the number of contributors; that is, $\frac{\overline{l+1}^2}{nn} + \frac{\overline{l+2}^2}{nn} + \frac{\overline{l+3}^2}{nn}$ &c. = number of contributors, the ſum of all which (by the differential method) is $= \frac{2nn-3n+1}{6nn}\, n - \frac{2ll+3l+1}{6nn}\, l =$ (putting $n-d$ for l) $\frac{\overline{n-d} \times \overline{d-1}}{n} + \frac{2dd-3d+1}{6nn}$ $d = \frac{\overline{n-d} \times \overline{d-1}}{n} + \frac{2d-3}{6nn}\, dd$ nearly.

5. In

5. In like manner for the widows, the number for d in the ſeries A, is $\frac{dl}{n}$. For $d-1$, in ſeries B, is $\frac{\overline{d-1}\cdot\overline{l+1}}{n}$. For $d-2$, in ſeries C, is $\frac{\overline{d-2}\cdot\overline{l+2}}{n}$; for $d-3$, in ſeries D, is $\frac{\overline{d-3}\cdot\overline{l+3}}{n}$ &c. for $d-1$ terms, beginning at B. The ſum of which (by the differential method) divided by n, is $\frac{dl}{nn}\times\overline{d-1}+\frac{\overline{d-1}\cdot\overline{d-1}}{2nn}d-\frac{2dd-3d+1}{6nn}=$ (putting $n-d$ for l) $\frac{\overline{d-1}}{2n}d-\frac{2d-3}{6nn}dd$, for the number of widows.

6. But as no ſupply has been made for the people dying in B, C, D, &c. which in n years are reduced to half. Therefore multiply each quantity by $1+\frac{d}{n}$, and putting $P=\frac{2d-3}{6nn}dd$, then we have $\frac{n+d}{n}\times:\frac{\overline{n-d}\cdot\overline{d-1}}{n}+P:$ for the number of contributors. And $\frac{n+d}{n}\times:\frac{\overline{d-1}}{2n}d-P:$ for the number of annuitants; excluſive of the ſeries A. And the ſame in proportion for any number greater than n.

7. Therefore if N be the number of couples in this ſociety. The number of contributors at the end of d years, will be $\frac{ll}{nn}N+\frac{n+d}{nn}N\times:\frac{\overline{n-d}\cdot\overline{d-1}}{n}+P:$

And

And the number of annuitants upon the list, will be $\frac{dl}{nn} N + \frac{n+d}{nn} N \times : \frac{d-1}{2n} d - P$: nearly, including the original number (A).

8. And after n years, the constant number of contributors, will be $\frac{2}{3} N$. And the constant number of annuitants, $\frac{1}{3} N$, nearly.

ART.

Fig.

ART. IV.

OF THE

MOON'S MOTION.

PROB. I.

To find the abſolute Force whereby the Moon is drawn from its Orbit, by the Action of the Sun; ſuppoſing the Moon's Orbit nearly circular.

LET QAZ be the moon's orbit, P the place of the moon at any time. Let the ſun's diſtance $= r$, moon's diſtance $PT = a$, ſine $PK = y$, arch $QP = z$, verſed ſine $QK = x$, $p =$ earth's periodical time in ſeconds $= 31556940$, $g =$ force of gravity, $b =$ ſpace deſcended in a ſecond by gravity, $\pi = 3.14159265$. 1.

The arch deſcribed by the earth in a ſecond is $\frac{2\pi r}{p}$, and $\frac{4\pi\pi rr}{2rpp}$ or $\frac{2\pi\pi r}{pp}$ is its verſed ſine, and this is the ſpace through which the earth is drawn by the ſun in a ſecond, which meaſures the ſun's force. By Cor. 1, Prop. 27, centr. forces, the ſun's diſturbing force at $P = \frac{2y}{r} C = \frac{2y}{r} \times \frac{2\pi\pi r}{pp}$ $= \frac{4\pi\pi y}{pp}$; and this is the difference of the forces at K and P; and therefore $\frac{4\pi\pi y}{pp}$ is the force that draws the moon from her orbit, directly towards the ſun, or rather the ſpace through which it is drawn from its orbit in a ſecond. Hence, if F be the

Fig. the force required, then $b : g :: \frac{4\pi\pi y}{pp} : F = \frac{4\pi\pi y}{ppb} g$

$= \frac{y}{b} \times \frac{g}{252249336180000}$.

Cor. If s be the ſine of QTP *to the radius* 1, *then*

$y = sa$, *and* $F = \frac{4\pi\pi sa}{ppb} g = \frac{sg}{21432.5b}$.

PROB. II.

1. *To find how far the moon is drawn from its orbit, in any time, from the quadrature, by the ſun's diſturbing force, ſuppoſing her orbit nearly circular.*

Let QPZ be her original orbit, P her place. Suppoſe a body P revolving in the orbit QPZ, as the moon would do if it was not diſturbed by the ſun. And ſuppoſe the line PS always drawn from the body P to the ſun; and that the moon is drawn away in the *line of vagation* PS to ☽, having deſcribed P ☽ in the time of moving from Q to P. To find the length P ☽; and the velocity at ☽, along the line PS.

Denoting the quantities as before, alſo putting $t =$ the moon's ſynodic time in ſeconds, P ☽ $= w$. Then $\frac{2\pi a}{t}$ = arch of the moon's orbit deſcribed in a ſecond. And $2\pi a : t :: \dot{z} : \frac{t\dot{z}}{2\pi a}$ = time of paſſing through $\dot{z}$. But (by Prob. I, Section III. Fluxions) $\dot{v} \propto F\dot{t}$ (v being the velocity at ☽), and in the caſe of falling bodies, $\dot{v} = 2b\dot{t}$, and $F = b$. Therefore $2b\dot{t} : b\dot{t} :: \dot{v} : \frac{4\pi\pi y}{pp} \times \frac{t\dot{z}}{2\pi a}$,

or

or $2 : 1 :: \dot{v} : \frac{2\pi ty\dot{z}}{app}$, therefore $\dot{v} = \frac{4\pi t}{app} \times y\dot{z} = \frac{4\pi t}{app} \times a\dot{x}$, by nature of the circle. Therefore $v = \frac{4\pi tx}{pp} = \frac{x}{31059503}$, or $v = \frac{a}{31059500} \times$ verſed ſine of the moon's diſtance from the laſt quadrature, radius $= 1$. And ſuppoſing V $=$ that verſed ſine, $a = 1176948000$; then $v = 37.8934$V.

Again, (Fluxions, *ib.*) $s \propto vt$, and in falling bodies $\dot{s} = 2b\dot{t}$, and $v = 2bt$; therefore $2bt : 2b \times \dot{t} :: \dot{w} : \frac{4\pi tx}{pp} \times \frac{t\dot{z}}{2\pi a}$. Therefore $\dot{w} = \frac{2ttx\dot{z}}{app} = \frac{2ttx}{app} \times \frac{a\dot{x}}{y} = \frac{2tt}{pp} \times \frac{x\dot{x}}{y} = \frac{2tt}{pp} \times \frac{x\dot{x}}{\sqrt{2ax - xx}} = \frac{4tt}{ppa} \times \frac{ax\dot{x}}{2\sqrt{2ax - xx}}$. But the fluent of $\frac{ax\dot{x}}{2\sqrt{2ax - xx}}$ is known to be $=$ ſegment QHP. Therefore $w = \frac{4tt}{ppa} \times$ ſegment QHP. Or $w = \frac{4tta}{pp} \times$ ſegment QHP, to the radius $1 = \frac{a}{38.2436255} \times$ ſegment QHP, radius $= 1$.

Cor. 1. *Hence in the ſyziges at* A, *the moon is drawn from its orbit, with a velocity of* 37.894 *feet in a ſecond. And at the next quadrature with a velocity of* 75.788 *feet, where the force nearly ceaſes.*

Cor. 2. *And in the ſyziges ſhe has departed* 9402985 *feet, or* 1780.87 *miles from her orbit. And at the next quadrature* 51752450 *feet, or* 9801.6 *miles, which is about* $1\frac{1}{4}$ *the earth's diameter, or* $4\frac{1}{2}$ *of the moon's, amounting to about* 4° 42′.

Cor.

Fig. Cor. 3. *The moon's perp. diſtance from its true orbit in any place, is as w × ſine of the arch* QP.

For the perp. diſtance of ☽ from the orbit PA is P☽ × ſine of the angle ☽PA or QTP.

SCHOL.

1. Strictly ſpeaking, the moon perhaps may not always keep in the line PS, but fall a little back, or be a little forward, in different parts of the orbit. But this cannot be certainly known, for want of knowing the original orbit.

2. Beſides the line P☽ deſcribed by the moon, has been here ſuppoſed to be a right line, upon ſuppoſition, that the ſun ſtands ſtill, and has no motion in longitude. But ſince in the time of half a ſynodic revolution, the ſun has moved forward about 14½ degrees, therefore the line deſcribed by the moon, moving towards the ſun, will be in the form PBM; of ſuch a nature that the extreme tangents P☽, MC will interſect at D in an angle of 14½°. And the radius of curvature at any point B, will be as the velocity of the moon at B, that is, as the verſed ſine of the correſpondent arch QH. Suppoſing PB deſcribed in the time QH is deſcribed by P.

After a like manner the velocity and ſpace deſcribed, along the curve QP, or along the radius TP may be calculated. And ſince I have performed the calculations, by a method ſomewhat different, I ſhall here ſet down the reſult thereof.

The velocity gained,

along the curve at A	—	20.32 feet.
in the radius at A	—	32 04
in the line PS at A	—	40.64

The

Fig.

The ſpace gained,

in the cure at A — $\frac{1}{1024}$ of TP.
in the radius TP at A — $\frac{1}{2048}$ of TP.
in the line PS at A — $\frac{1}{1024}$ of TP.

PROB. III.

To find the place where the moon is furtheſt from the original orbit, in the line perpendicular to it. 3.

Let O be the moon's place in the time of vagation KPO, draw TP ☽ from the center T, and O ☽ parallel to the orbit at P. Let QK $= x$, KP $= y$, PO $= w$, QT $= 1$, then TK $= x - 1$. Then $1 : y :: w : P☽ = wy =$ maximum, and $\dot{y}w + w\dot{y} = 0$, that is, $\frac{2tt}{pp} \times x\dot{x} + \frac{4tt\dot{y}}{pp\alpha} \times$ ſeg. QHP $= 0$, and $x\dot{x} + \frac{2\dot{y}}{\alpha} \times$ ſeg. QHP $= 0$. Or $x\dot{x} + \frac{2\alpha\dot{x} - 2x\dot{x}}{\alpha y} \times$ ſeg. QHP $= 0$, or $\alpha y x + \overline{2\alpha - 2x} \times$ ſeg. QHP $= 0$, or $\frac{1}{2} \alpha x = \frac{x - \alpha}{y} \times$ QHP; put $\alpha = 1$, therefore $\frac{\frac{1}{2}xy}{x - 1} =$ ſeg. QHP $= \frac{1}{2}$ arch QHP $- \frac{1}{2}y$. And arch QHP $= y + \frac{xy}{x - 1} =$ (putting TK $= c$) $y + \frac{\overline{1 + c}.y}{c} = 2y + \frac{y}{c} =$ tan. ZP $+ 2$ ſine of ZP. Whence arch QP $= 135°$ $46'$ nearly. And then P ☽ $= \frac{a}{59}$ the greateſt diſtance from her orbit, and Po $= \frac{a}{41}$.

Or

Fig.

Or thus.

4. Draw ☽ATD and ☽PKG, and put $x = ☽A$, then rect. $A☽D$ = rect. $P☽G$, or $xx + 2ax = w^2 + 2yx$, and x is to be a maximum, put m for x, then $m^2 + 2am = w^2 + 2yw$, in fluxions $2w\dot{w} + 2y\dot{w} + 2w\dot{y} = 0$; that is, $\overline{w+y} \times \frac{x\dot{x}}{y} + \frac{2\text{ſeg.QHP}}{a} \times \frac{a\dot{x} - x\dot{x}}{y} = 0$, and $wx + yx + \frac{2\text{ ſeg. QHP}}{a} \times \overline{a - x} = 0$, and $\frac{\frac{1}{2}ayx}{x-a} = \text{ſeg. QHP} - \frac{wxa}{2.\overline{x-a}}$ $= \text{ſeg. QHP} - \frac{2\pi x}{pp.\overline{x-a}} \times \text{ſeg. QHP}$; where, if the laſt term be neglected, as being very ſmall, it comes to the ſame as the firſt.

Put $a = 1$, $x - 1 = c$, then $\frac{\frac{1}{2}axy}{x-a} = \frac{1}{2} \times \frac{1+c}{c}$ $y = \frac{1}{2}\frac{y}{c} + \frac{1}{2}y = \frac{\text{tan. ZP} + \text{S.ZP}}{2} = \text{ſeg QHP} - \frac{1+c}{c} \times \frac{\text{ſeg. QHP}}{76\,4872}$. For ſeg. QHP, put arch $\frac{QP - PK}{2} = \frac{QP - y}{2}$, and proſecuting the calculation, there will be found x or $☽A = .0153 = \frac{a}{65.5}$, and $ZP = 43° 33'$, or $QP = 136° 27'$, $PA = 1° 6'$, $P☽$ or $w = .02213 = \frac{a}{45}$.

PROB.

Fig. 4.

PROB. IV.

To find the area QP ☽, *described by the line of vagation* P ☽.

The fluxion of the area QP ☽ is P ☽ × $\dot{x}$ = $w\dot{x}$ = $\frac{4tt\dot{x}}{ppa}$ × segment QHP. But segment QHP = $\frac{za-ya}{2}$; therefore the flux. area = $\frac{4tt\dot{x}}{ppa} \times \frac{za-ya}{2}$ = $\frac{2tt}{pp} \times \overline{z\dot{x}-y\dot{x}}$. But fluent of $y\dot{x}$ = area QKP, and let the fl. $z\dot{x}$ = $zx+s$, then $z\dot{x}$ = $z\dot{x}+x\dot{z}+\dot{s}$, and $\dot{s}$ = $-x\dot{z}$ = $-\frac{ax\dot{x}}{y}$ = $-\frac{ax\dot{x}}{\sqrt{2ax-xx}}$, and therefore s or the fl. $\frac{ax\dot{x}}{\sqrt{2ax-xx}}$ = 2 area QHP. Therefore the area QP ☽ = $\frac{2tt}{pp} \times$: zx + 2 area QHP − area QKPH : = $\frac{2tt}{pp} \times$: zx + seg. QHP − triangle QKP : = $\frac{2tt}{pp} \times$: $zx + \frac{\overline{z-y} \cdot a-yx}{2}$: where $\frac{2tt}{pp}$ = $\frac{1}{76.48725}$.

Cor. 1. *In the syzigies at* A, *the area will be* = $\frac{3}{4}aa \times 3.1416 - aa$: $\times \frac{2tt}{pp} = \frac{aa}{56.4}$.

Cor. 2. *At the next quadrature* Z, *the area* = $\frac{2tt}{pp} \times 2\frac{1}{2}aa \times 3.1416 = \frac{aa}{9.74}$.

Fig.

PROB. V.

To find nearly the original orbit of the moon.

1. We found in Prop. II. how far the moon is drawn from its orbit, at any place therein, or in deſcribing any arch. But ſince the time of deſcribing that arch is proportional to the ſynodic revolution; that arch muſt be reduced to the periodical time. Therefore in the quadrant QA=90°, ſay as the ſynodic, to the periodic time; ſo is 90°, to 83° 16′. The area of the ſegment QHP for 83° 16′ is .23011; whence the line of vagation P D is = $\frac{1}{166.18}$ *a*.

5. Let QAZ be the preſent orbit. Make QA = 90°, and draw AT, and ſet $\frac{1}{166}$ TQ from A to *a*, and the ſemi-circle Q*a*N being drawn will be nearly the original orbit, or that deſcribed without perturbation, wherein ZN = $\frac{1}{83}$ TZ. And TZ : TN : : 83 : 82. The ſame contraction will be made in the other half of the orbit. And therefore the diameter of the preſent orbit, is to the diameter of the original orbit, as 83 to 82.

If the arch of 90 had not been reduced to 83 16, the diameter of the diſturbed orbit, would be to the diameter of the undiſturbed orbit, as 67 to 66.

Cor. *Hence, by the action of the ſun, the moon's orbit is lengthened at leaſt* $\frac{1}{82}$ *part of itſelf; and the periodical time is increaſed accordingly; which will be* $\frac{1}{83}$ *part, or near* 12 *hours more.*

SCHOL.

From the foregoing problems, it appears how much the moon's motion is diſturbed, and her orbit diſtorted, merely by the ſun's variable force acting

ing upon her. From which ſo many irregularities ariſe, both in her place and diſtance; that no tables have ever been made, to anſwer truly for any conſiderable length of time. Fig.

If her undiſturbed orbit and periodical time in it were given; and the time of the moon's being in the quadrature known; I doubt not but one might find the moon's place very near, by the ſecond Prob. for a given time; and be able to compoſe tables to find her place from any time till the next quadrature, but no longer. For after that ſhe will be put into a new tract, which being diſturbed over again, will differ from the former, and after that into another ſtill different, and ſo on. So that after every ſemi-revolution ſhe will be put into a new path different in ſome degree from the former. Yet in time ſhe may deſcribe orbits which will converge to ſome regularity. But I have neither time nor leiſure to proſecute theſe things any further.

The action of the ſun on the further ſide of the moon's orbit is ſomething leſs than that on the neareſt ſide, which alſo cauſes ſome error. But then the moon (as all the other ſatellites do) moving the ſame way about the earth, as the earth moves about the ſun; the centrifugal force of the moon is greater on the outſide, which compenſates for the leſs force of the ſun. For on the outſide of the orbit, the moon's abſolute motion is the ſum of the motions of the earth and moon, in their orbits; but on the inſide it is their difference; and the centrifugal forces are as theſe motions.

ART. V.

The Conſtruction of the Arches of Bridges, &c.

UPON account of the great uſe, importance, and curioſity of this ſubject, I took the pains, in my book of *Fluxions*, to calculate the nature of the curve, upon which all the parts of an arch will be in equilibrio. And in the *Mechanics*, I have given numbers for the conſtruction of ſuch arches. For no architect or engineer that knows what he is about, will venture to build any arches, but what ſhall be the ſtrongeſt and moſt durable that can be made. And that can only be when all the parts are in an exact equilibrium; ſo that they cannot thruſt out one another, and fall down. And here it is ſuppoſed the ſtones have no coheſion with one another, but are entirely ſuſtained by the figure of the arch. So that if they were laid in oil, and could eaſily ſlide by one another, they would ſtill keep their places. But being laid in lime or mortar to make them cohere, they could then bear any additional weight that could come upon them. I ſhall, in what follows, give the inveſtigation of the numbers by which the curve is to be drawn, together with ſome other things relating to the ſubject.

PROP.

Fig.

PROP. I. PROB.

Several inflexible right lines, moveable about the points B, F, H, L, O, *being given in a vertical plane; but the extreme points* A, R, *fixt immoveable. To find the proportion of the weights or forces inſiſting on the angles* B, F, H, *&c. ſo that the figure may ſtand in equilibrio, in that poſition.* 6.

Through the angles B, F, H, &c. draw perpendiculars to the horizon, B*c*, F*e*, H*i*, &c. which are the direction of the weights or forces; and taking B*c* of any length compleat the parallelogram *a*B*bc*; likewiſe take *d*F = B*b*, and compleat the parallelogram *d*F*fe*. In like manner make *g*H = F*f*, *k*L = *b*H, *n*O = L*l*, compleating all the parallelograms *g*H*bi*, *k*L*lm*, *n*O*op*. Then the forces or weights on the angles B, F, H, L, O, are reſpectively as the diagonals B*c*, F*e*, H*i*, L*m*, O*p*.

For the force B*c* is equivalent to B*a* and B*b*. B*a* acts towards the immoveable point A, and B*b* acts towards F. Alſo the force F*e* is reſolved into the forces F*d* and F*f*. But F*d* acts towards B, and is oppoſed and kept in equilibrio by the equal force B*b*, becauſe B*b* = *d*F, by conſtruction. After the ſame manner the force H*i* is reſolved into H*g*, H*b*: the force L*m* into L*k*, L*l*: the force O*p* into O*n*, O*o*. Of theſe the forces F*f*, H*g*; and H*b*, L*k*; and L*l*, O*n*, deſtroy one another, being equal and contrary; and the force O*o* acts againſt the immoveable point R; and therefore all the points B, F, H, L, O, will remain unmoved.

After the ſame manner may the forces be found, when they act not perpendicular to the horizon, but tend towards a given point; and that is, by drawing the lines B*c*, F*e*, H*i*, &c. to that point, and

Fig. 6. compleating the parallelograms, and proceeding as before.

Now the force on any point B, is to the force on any other point O : : B*c* to O*p*. But B*c* to O*p*, is in the complicate ratio of

$$\left.\begin{matrix} \text{B}c \text{ to B}b \text{ or } d\text{F} \\ d\text{F} \text{ to } de \text{ or } g\text{H} \\ g\text{H} \text{ to } gi \text{ or L}k \\ \text{L}k \text{ to } km \text{ or } n\text{O} \\ n\text{O} \text{ to O}p \end{matrix}\right\} \text{that is,} \left\{\begin{matrix} cab : c\text{B}a \\ e\text{FH} : e\text{FB} \\ i\text{HL} : i\text{HF} \\ m\text{LO} : m\text{LH} \\ p\text{OR} : \text{ROL} \end{matrix}\right\}$$

But *s.e*FH, and *s.i*HF are ratio's of equality; and *s.i*HL and *s.m*LH are ratio's of equality. Therefore B*c* : O*p* : : *s.cab* × *s.m*LO × *s.p*OR : *s.c*B*a* × *s.e*FB × *s.*ROL : : *s.*ABF × *s.p*OL × *s.p*OR : *s.c*BA × *s.c*BF × *s.*ROL : : $\frac{s.\text{ABF}}{s.c\,\text{BA} \times s.c\,\text{BF}}$: $\frac{s.\text{ROL}}{s.p\text{OL} \times s.p\text{OR}}$: : force on B ; force on O.

Cor. 1. *The forces on any two angles* B. L, *in directions* BA, LO; *are directly as the secants, or reciprocally as the cosines of elevations, that is, as s.m*LO *to s.c*BA.

For the forces B*a* and L*l*, are in the complicate ratio of

$$\left.\begin{matrix} \text{B}a \text{ to B}c \\ \text{B}c \text{ to L}m \\ \text{L}m \text{ to L}l \end{matrix}\right\} : : \begin{matrix} s.c\text{BF} : s.\text{ABF}. \\ \frac{s.\text{ABF}}{s.c\text{BA} \times s.c\text{BF}} : \frac{s.\text{HLO}}{s.m\text{LH} \times s.m\text{LO}} \\ s.\text{HLO} : s.m\text{LH}. \end{matrix}$$

$$: : \frac{1}{s.c\text{BA}} : \frac{1}{s.m\text{LO}}.$$

Cor. 2. *The weights or forces upon the angles will continue the same; if the inclinations of the lines to one another, and to the horizon, continue the same; though the lengths of the lines be never so much varied.*

For

For the lengths of the lines do not come into the computation. Fig. 6.

Cor. 3. *The direction of the force on any point* F *is in the line* FB *downwards, and in the line* FH *upwards.*

This appears by the conſtruction.

Cor. 4. *The force inſiſting on any angle* F *is as the ſum of the cotangents of* eFB, eFH, *when they are both acute; but as the difference, when one is obtuſe.*

For the force on F is as $\frac{s.BFH}{s.eFB \times s.eFH}$. But $s.BFH = s.BFe \times \text{cof. } eFH + s.eFH \times \text{cof. } BFe$. Therefore the force on F is as $\frac{s.BFe \times \text{cof. } eFH + s.eFH \times \text{cof. } BFe}{s.eFB \times s.eFH} = \frac{\text{cof. } eFH}{s.eFH} + \frac{\text{cof. } BFe}{s.BFe} = \text{cot. } eFH + \text{cot. } BFe$. But here the angle eFH being obtuſe, the coſine and cotan. will be negative; therefore the force at F is as cotan. BFe — cotan. eFH.

Cor. 5. *If any external angle* ABx *be infinitely ſmall, then any weight preſſing at* B *will act as much againſt* F *as againſt* A.

For if ABx or Bac be infinitely ſmall, *ac* coincides with *a*B; and the forces in directions BF, BA (being as the equal lines *ac*, *a*B) will be equal.

PROP. II.

If a weight is ſuſtained by the curve AFV, *in a vertical plain; and is kept in equilibrio by the weight inſiſting on every point. Then the weight or preſſure upon any point* F *is as the curvature at* F *directly, and the ſquare of the ſine of the angle* eFy *or* TFI *reciprocally:* TFy *being a tangent, and* TI *perp. to the horizon.* 7.

For draw the external angle BFy by producing HF (Fig. 6.) and ſuppoſe all the lines AB, BF, FH,

Fig. 7. FH, &c. to be equal. Then (by Prop. I.) the force or preſſure on any angle F, is as $\frac{s.BFH}{s.BFe \times s.eFH}$. Now let the number of the equal lines be augmented, and their length diminiſhed ad infinitum, and the figure AFOR will become a curve line; and the preſſure will act on every point of the curve; and the infinitely ſmall evaneſcent angle BF*y* becomes the angle of contact, and *e*FB becomes $=$ *e*F*y*; and the preſſure at F being as $\frac{s.BFH}{s.BFe \times s.eFH}$ or $\frac{s.BFy}{s.eFB \times s.eFy}$ is $= \frac{s.BFy}{s.eFy^2}$. But when the infinitely ſmall part of the curve is given, the ſine of the angle of contact BF*y* is as the curvature. Therefore (in Fig. 7.) the force on F is as the curvature at F directly, and the ſquare of the ſine of *e*F*y* or ITF reciprocally.

8. Cor. 1. *If a weight like a wall* APQR *be incumbent on the arch* AFR, *ſtanding in a vertical plane, and all the parts kept in equilibrio; then I ſay the height* Fg *on any point* F, *is as the curvature at* F *directly, and the cube of the ſine of* ITF *reciprocally.*

For the weight on the given part of the curve Fr is as $\frac{C}{s.ITF^2}$, C being the curvature at F, and the weight of the column Fg*br* is as $Fg \times wr = Fg \times Fr \times s.FTI = Fg \times s.FTI$, becauſe Fr is given. Therefore $Fg \times s.FTI$ is as $\frac{C}{s.FTI^2}$, and Fg as $\frac{C}{s.FTI^3}$, to keep the parts in equilibrio.

Cor. 2. *If* R *be the radius of curvature in* F, *then the height* Fg *is as* $\frac{1}{R \times s.FTI^3}$.

Cor.

Cor. 3. *If* ZI $= x$, FI $= y$, ZF $= z$; *then*, Fg *is as* $\frac{-x\ddot{y}}{\dot{y}^3}$, *where* $\dot{x}$ *is constant.* Fig. 8.

For $\dot{x} : \dot{y} :: FT : TI :: rad. 1 : s.FTI = \frac{\dot{y}}{\dot{z}}$,

therefore $Fg = \frac{\dot{z}^3}{R \times \dot{y}^3}$. And R may be found by supposing either $\dot{x}$ or $\dot{y}$ given. If $\dot{x}$ be given, $R = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$. Therefore Fg is as $\frac{\dot{z}^3}{\dot{y}^3} \times \frac{-\dot{x}\ddot{y}}{\dot{z}^3} = \frac{-\dot{x}\ddot{y}}{\dot{y}^3}$.

In my book of Fluxions I have calculated the height Fg, for several curves by the method of Fluxions, by Cor. 3. I shall in the next Prop. do it by Cor. 2. for the sake of such as do not understand Fluxions.

PROP. III. PROB.

The nature of the curve AFZR *being given for the figure of an arch; to find the height* Fg, *of the wall insisting thereon, at every point* F; *so that all the parts shall remain in equilibrio.* 8.

Draw the ordinate FI, and let ZI $= x$, FI $= y$, ZF $= z$, ZS $= a$, R $=$ rad. curvature in F, draw the tangent FT, which will be found from the nature of the curve. Find the S. angle FTI; then take Fg as $\frac{1}{R \times s.FTI^3}$, for the height.

EXAM. 1.

Let FZ be the arch of a circle, radius $= r$, ZI $= x$, cos. FZ $= c$, ZS $= a$, then $c = s.FTI$, whence Fg is as $\frac{1}{rc^3}$ or as $\frac{1}{c^3}$, which at Z is $\frac{1}{r^3}$. Therefore

$\frac{1}{r^3} : \frac{1}{c^3} :: a : Fg = \frac{ar^3}{c^3}$.

And

Fig. And ſubtracting ZI or $x = r - c$, then ZS = $\frac{ar^3}{c^3} - \overline{r-c}$, for the height in any place F, above Z.

EXAM. 2.

9. Let the curve FZ be an ellipſis, t = AC the ſemi-tranſverſe parallel to the horizon, c = ZC the ſemi-conjugate, ZI = x, FI = y, ZS = a, π = FQ perp. to the tangent FT. Then (Ellipſis, Prop. XI. Cor. 1.) $cc : tt :: \overline{c-x}$ (CI) : $\frac{tt}{cc} \times \overline{c-x}$ = QI. And FQ (π) : rad. (1) :: QI : S.QFI or FTI = $\frac{tt \times \overline{c-x}}{cc\pi}$. And rad. curvature at F = $\frac{4\pi^3}{bb}$, b being the parameter (to CZ) = $\frac{2tt}{c}$. Therefore $\frac{1}{R \times \text{S.FTI}^3} = \frac{bb}{4\pi^3} \times \frac{c^6\pi^3}{t^6 \times \overline{c-x}^3} = \frac{4t^4}{4cc\pi^3} \times \frac{c^6\pi^3}{t^6 \times \overline{c-x}^3} = \frac{c^4}{tt \times \overline{c-x}^3}$, which is as Fg; whence QZ = $\frac{tt}{c}$, and ZS $\left(\frac{c^4}{ttc^3}\right) : a :: \frac{c^4}{tt \times \overline{c-x}^3} : \text{Fg} = \frac{c^3a}{\overline{c-x}^3}$. Alſo $\frac{c^3a}{\overline{c-x}^3} - x$ = the height of g above Z.

EXAM. 3.

10. Let AFZ be another ellipſis, whoſe ſemi-tranſverſe CZ is perp. to the horizon, and ſemi conjugate AC equal to the ſemi-tranſverſe of the other. Let AC = t, ZC = n, ZI = x, FI = y, and equal to the former FI. ZS = a, π = FQ perpen. to the tangent FT. Then as before, the parameter $b = \frac{2tt}{n}$, and QI = $\frac{tt}{nn} \times \overline{n-x}$, and S.FTI =

$tt \times$

Fig. 10.

$\frac{tt \times \overline{n-x}}{nn\pi}$, and the radius of curvature at F $=$ $\frac{4\pi^3}{bb} = \frac{t^4}{nn\pi^3}$; therefore $\frac{1}{R \times S.FTI^3} = \frac{n^4}{tt \times \overline{n-x}^3}$ is as Fg, and Fg $=$ $\frac{n^3 a}{\overline{n-x}^3}$.

And $\frac{n^3 a}{\overline{n-x}^3} - x =$ height of g above Z.

SCHOL.

From these three examples it will appear what is the comparative strength of these three arches, the flat ellipsis, the circle, and the sharp ellipsis, having all of them the same breadth, and the same height ZS at the top Z. For taking FI the same in all; then in the flat ellipsis (Fig. 9.) $tt : cc :: yy : cc - zz$, and $zz = cc - \frac{ccyy}{tt}$ and $\frac{zz}{cc} = 1 - \frac{yy}{tt} = \frac{tt-yy}{tt}$, and $\frac{cc}{zz} = \frac{CZ^2}{CI^2} = \frac{tt-yy}{tt}$, or Fg $=$ $\frac{CZ^3}{CI^3} = \frac{tt-yy}{t^3}\sqrt{tt-yy}$, where t and y being always the same, the height Fg is precisely the same, in all the three figures.

But if the top of the wall is bounded by a horizontal right line passing through S, the height then will be $Fg - a - x = b - x$. Therefore when x is greatest, the height above S is the least, which is in the sharp ellipsis, and in this figure the want of balance at F is the least; and in the flat ellipsis it is the greatest. Therefore upon that account, the sharp ellipsis is strongest, the circle next, and the flat ellipsis weakest.

EXAM.

Fig. 8.

EXAM. 4.

Let FZ be a parabola, $rx = yy$, the ſubtangent $IT = 2x$, and $FT = \sqrt{yy + 4xx} = \sqrt{rx + 4xx}$, FT : FI : : rad. (1) : S.FTI, that is, $\sqrt{rx+4xx}$: $\sqrt{rx}$: : 1 : $\sqrt{\frac{rx}{rx+4xx}} =$ S.FTI, and rad. curvature at F $= \frac{\overline{rr+4rx}^{\frac{3}{2}}}{2rr}$, whence Fg $\left(\frac{1}{R \times \overline{S.FTI}^3}\right)$ is as $\frac{2rr}{\overline{rr+4rx}^{\frac{3}{2}}} \times \frac{\overline{rx+4xx}^{\frac{3}{2}}}{\overline{rx}^{\frac{3}{2}}} = \frac{2rr}{r^3}$ a given quantity. And ſince ZS $= a$, the height every where is equal to a.

I omit the hyperbola, becauſe the curve PSQ, continually approaches to the hyperbola, and therefore is unfit for an arch.

EXAM. 5.

Let FZ *be the catenary.* Then by the nature of the curve $zz = 2rx + xx$, and ſubtangent $IT = \frac{zy}{r}$, and $FT = \sqrt{yy + \frac{zzyy}{rr}}$, and $\sqrt{yy + \frac{zzyy}{rr}} : y : :$ rad. 1 ; S.FTI $= \frac{y}{\sqrt{yy + \frac{zzyy}{rr}}} = \frac{1}{\sqrt{1 + \frac{zz}{rr}}} = \frac{r}{\sqrt{rr+zz}}$; and the rad. curvature at F $= \frac{rr+zz}{r}$. Whence Fg $\left(\frac{1}{R \times \overline{S.FTI}^3}\right)$ is as $\frac{r}{rr+zz} \times \frac{\overline{rr+zz}^{\frac{3}{2}}}{r^3} = \frac{\sqrt{rr+zz}}{rr}$; and in Z it is $\frac{1}{r}$, therefore

$\frac{1}{r}$

$\frac{1}{r} : a :: \frac{\sqrt{rr+zz}}{rr} : Fg = \frac{a\sqrt{rr+zz}}{r} = \frac{a \times \overline{r+x}}{r}$ Fig. 8.

$= a + \frac{ax}{r}$.

Cor. 1. *If* $a = r$, *then* Fg *is every where* $=$ IS, *and* PSQ *is a right line.*

Cor. 2. *If a be greater than r,* Fg *will be greater than* IS; *and therefore the curve* PS *runs upwards, and is convex towards* Z.

Cor. 3. *If a be leſſer than r, then* Fg *will be leſſer than* IS, *and the curve* PS *runs downwards, or is concave towards* Z.

Moreover if a be very ſmall, the perp. thickneſs of the arch is the ſame every where.

For if p be the perp. thickneſs of the arch; then $Fg : p ::$ rad. $:$ S.FTI $::$ FT. $:$ FI; that is, FT $\left(\sqrt{yy + \frac{zzyy}{rr}}\right)$: FI $(y) ::$ Fg $\left(\frac{\overline{r+x} \times a}{r}\right) : p =$

$\frac{ay \times \overline{r+x}}{r\sqrt{yy + \frac{zzyy}{rr}}} = \frac{a \times \overline{r+x}}{\sqrt{rr+zz}} = \frac{a \times \overline{r+x}}{\sqrt{rr + 2rx + xx}} =$

$\frac{a \times \overline{r+x}}{r+x} = a$: conſequently a heavy flexible line put into this figure would ſupport itſelf.

EXAM. VI.

Let FZ be the logarithmic curve, HD the aſymptote, FT a tangent at F. ZD $= r$, ſubtangent HR $= t$, which is always the ſame. Then FH $= r + x$, FR $= \sqrt{tt + rr + 2rx + xx}$, then FR $(\sqrt{tt + rr + 2rx + xx})$; HR $(t) ::$ rad. $(1) :$ S.HFR or FTI $= \frac{t}{\sqrt{tt+r+x}}$. And the rad. cur- 112

vature

Fig. 11. vature at F $= \frac{\overline{tt+FH^2}^{\frac{3}{2}}}{t \times FH} = \frac{\overline{tt+r+x^2}^{\frac{3}{2}}}{t \times \overline{r+x}}$. Therefore Fg is as $\frac{t \times \overline{r+x}}{\overline{tt+r+x^2}^{\frac{3}{2}}} \times \frac{\overline{tt+r+x^2}^{\frac{3}{2}}}{t^3} = \frac{r+x}{tt}$, which at Z is $\frac{r}{tt}$, therefore $\frac{r}{tt} : a :: \frac{r+x}{tt} : Fg = \frac{a}{r} \times \overline{r+x}$, the same as for the catenary.

Cor. *Hence if* $a = r$, *then* Sg *becomes the asymptote* DH, *and is therefore a right line.*

If a is less than r, *the curve* Sg *is concave towards* Z, *descending towards g. If a be greater than r, it is convex towards* Z, *rising upwards.*

PROP. IV.

12. *If there be two arches of equilibration of the same kind,* AFZR *and* AfzR, *upon the same base* AR; *the strength of the high arch is to the strength of the flat arch throughout; reciprocally as the radius of curvature at z to that at* Z.

In the Schol. to Exam. 3, of the last Prop. I shewed the weakness of the *same arch* in different points, by an undue weight. Here I shew the comparative strength in *different arches* of the same kind, by which I mean those whose ordinates N*f*, NF are every where in a given ratio.

The materials whereof any bridge is made, are not of infinite strength, but there is a certain degree of pressure that will crush them to pieces, or thrust out the butment.

Let H (in Fig. 6.) be the vertex of the figure. In the demonstration of Prop. I. I have shewn, that if Hg, H*b* are the forces along the lines HF, HL,

HL, that the weight H*i* is the diagonal of the parallelogram *g*H*bi*. Now ſuppoſe *g*H, H*b*, the elements of a curve, and the arches *g*H, H*b*, to be given, being the greateſt forces they can ſuſtain; then H*i* is twice the verſed ſine of the arch *g*H, which repreſents the weight. Therefore $\frac{gH^2}{\frac{1}{2}Hi}$ = twice the radius of curvature = 2R, or $\frac{gH^2}{Hi}$ = R, whence R is reciprocally as the weight H*i*, or the weight H reciprocally as the radius of curvature at H. Therefore (in Fig. 12.) the weight at Z to the weight at *z*, is as rad. curvature at *z*, to rad. curvature at Z; which is all theſe bridges can ſuſtain. Fig. 12.

Cor. 1. *If* AFZ, A*fz* *be two ellipſes; their ſtrengths are as their heights* CZ *and* C*z*. *And ſo it is for the circle and ellipſis.*

For the radius of curvature at Z is $\frac{AC^2}{CZ}$, and the rad. curvature at *z* is $\frac{AC^2}{Cz}$; and theſe are as CZ to C*z*.

Cor. 2. *If* A*f*z*R* *be an ellipſis, the ſtrength of the arch at the vertex z, is no greater than that of a circle, whoſe radius is* $\frac{AC^2}{zC}$, *the radius of curvature in z.*

PROP. V.

In an arch of equilibration AFZR, *the preſſure at any point* F, *ariſing from the incumbent weight, is in direction of the tangent* TF*t*, *the tangent at* F. 13.

For (Fig. 6.) it was ſhewn in Cor. 3. Prop. I. that the direction of the force at F is in direction of

Fig. 13. of the line FB. And when the lines BF, FH, &c. are diminiſhed ad infinitum, and come into the form of a curve (Fig. 13); the direction of the preſſure is ſtill in direction of the particle of the curve at F, that is in the tangent TF*t*.

Cor. 1. *The weight of the part of the arch gz; the preſſure along the tangent* TF*t*, *and the lateral or horizontal preſſure in direction* IF; *are reſpectively as* TI, TF *and* IF.

For the weight acts in direction TI, and the preſſure in the tangent acts in direction TF, and the lateral preſſure acts againſt the plain TI in direction FI perp. to that plane. And theſe forces are as the three lines TI, TF and IF.

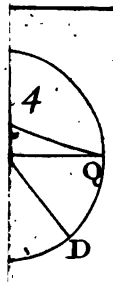
Cor. 2. *If the arch is not truly in equilibrio; and if there is a defect of preſſure at* F; *then the preſſure of the arch* ZF *at* F, *will be in a direction higher than the tangent* F*t*. *But if there is an exceſs of preſſure at* F; *the preſſure at* F *will be lower than the tangent* F*t*.

This follows from the compoſition of motion.

PROP. VI.

In any arch of a bridge, &c. the joints in every place F, *muſt be perpendicular to the curve of the arch.*

14. Every body acting againſt a plain ſurface, acts in lines perpendicular thereto; but if a body preſſes obliquely againſt a plane, it will endeavour partly to ſlide along that plane. And if two ſurfaces be preſſed together by an oblique force, theſe ſurfaces will endeavour, according to the degree of obliquity, to ſlide by one another. Therefore to prevent that effect, theſe ſurfaces muſt be preſſed together by a force which acts perpendicular thereto. Hence, if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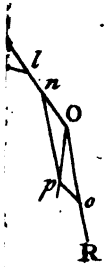
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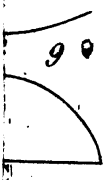
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if CF be a joint where the ſurfaces of two arch ſtones meet, and as BA is the direction of the force; therefore CF muſt be perp. to the arch AB, at F; ſo that AC and CB may remain at reſt. Fig. 14.

Cor. *Hence if the joints be not perpendicular to the curve, there will ariſe a lateral preſſure, whoſe direction is not along the tangent; and wanting a force to ſuſtain it, will deſtroy the equilibrium; and ſome of the ſtones will endeavour to fly out.*

SCHOL.

What has been ſaid of a linear arch, muſt be underſtood of arches of any thickneſs, or ſuch as are contained between two parallel plain arches; ſuch as we ſee in churches, bridges, &c. Theſe may be ſuppoſed to be generated by the motion of a linear arch, and its plane, in direction of a line perp. to that plane. Likewiſe what is ſaid of the height Fg, muſt be underſtood of the ſection of a plane paſſing through Fg, and perpendicular to the plane of the arch.

I muſt obſerve, that the tops of the arch-ſtones at C muſt be formed as you ſee in the figure, one part of each being parallel, and another part perpendicular to the horizon.

PROP. VII.

If AFZR *be an arch of equilibration, and* APQR *the incumbent weight. Then if all the ordinates* ZS, Fg, AP, RQ, *&c. be divided at* D, E, B, *b*, *&c. in any given ratio; and the arch b*DEB *be drawn through all the points of diviſion* D, E, *&c. Then the weight* ABD*b*R *will alſo keep the arch* AFZR *in equilibrio.* 15.

For the points at Z, F are kept in equilibrio, by the heights of the wall or arch at Z, F; that is,

Fig. 15. by ZS and Fg, and ZD, FE being in the ſame ratio as ZS and Fg, the ſame equilibrium muſt remain.

Or it may be conſidered thus; ſince all the parts of the wall keep all the points of the arch in equilibrio; let ZD, FE, be $\frac{1}{2}$ or $\frac{1}{3}$ &c. of ZS, Fg; and then let the wall DEFZ be twice or thrice, &c. as denſe matter, as SgFZ; then it is plain there will be the ſame preſſure upon every point as before; and the arch will ſtill be kept in equilibrio.

Cor. *Hence if the figure of the arch* AFZ *be given, and alſo the figure at top* SgP; *we can find another curve* DEB, *nearly reſembling any figure we pleaſe (if poſſible), which ſhall ſtill preſerve the equilibrium. And that is by dividing* ZS, Fg, *in ſuch a ratio at* D *and* E, *that the points* D, E, *may fall in the figure required.*

For an infinite number of curves DE may be drawn by ſuch a diviſion; and ſome or other of them will nearly repreſent the figure we want, when the thing is poſſible.

EXAM. I.

16. Let AFZ be a circle, Sg the curve at top, found by Ex. 1. Prop. II. To find the poſition of the horizontal right line DE, ſo that the weight ZDEF may keep the part of the circle ZF in equilibrio; for it is impoſſible to do it for the whole circle.

Let AF be 30 degrees, $CF = r$, $CI = c$, $ZS = a$, $ZD = z$. Then $Fg = \frac{ar^3}{c^3}$. Then $CI = IZ$, $ID = \frac{1}{2}r + z = FE$, whence $ZD : ZS :: FE$ or $ID : Fg$; that is, $z : a :: \frac{1}{2}r + z : \frac{ar^3}{\frac{1}{8}r^3}$ or $8a$; therefore $8az = \frac{1}{2}ar + az$, or $7z = \frac{1}{2}r$, whence $z = \frac{1}{14}r$.

EXAM.

Fig.

EXAM. 2.

Let the arch AZR *be the catenary whoſe latus rectum is r. To erect the wall* ABbR *upon it, ſo that all the parts of the arch may remain in equilibrio, and that the top or curve* BEDF *may have a proper degree of aſcent.* 17.

Make ZS = *r* in the axis CZ, and through S draw the horizontal line PSQ. Then the ſolid APQR will keep all the parts of the arch in equilibrio. Aſſume a point D at pleaſure, and make as ZS : ZD : : Fg : FE : : AP : AB. Draw BED for the top. If the aſcent through BED be too little or too great; it is too near PQ or AZ; and then a new point D muſt be taken nearer Z or S, reſpectively; and the points E, R, found anew; till at laſt the curve BED be ſuch as was required.

PROP. VIII. PROB.

To inveſtigate and conſtruct the curve or arch AFZ, *of equilibration; having given the points* A, Z, C, *and* ZS *the diſtance of the horizontal line* SP; *ſo that the weight* APSZ *may keep the arch* AZ *in equilibrio.* 18.

By Cor. 2. Prop. II. Fg is as $\frac{1}{R \times \overline{S.FTI}^3}$ (Fig. 8.) But $S.FTI = \frac{FI}{FT} = \frac{\dot{y}}{\dot{x}}$, and $R = \frac{\dot{x}^3}{\dot{y}\ddot{x}}$ when $\dot{y}$ is given; therefore $R \times \overline{S.FTI}^3 = \frac{\dot{x}^3}{\dot{y}\ddot{x}} \times \frac{\dot{y}^3}{\dot{x}^3} = \frac{\dot{y}^2}{\ddot{x}}$, and Fg is as $\frac{1}{R \times \overline{S.FTI}^3}$ or as $\frac{\ddot{x}}{\dot{y}^2}$.

Put ZS = *a*, ZI = *x*, FI = *y*, (Fig. 18.) then Fg or $a+x$ is as $\frac{\ddot{x}}{\dot{y}^2}$, or $a+x = \frac{rr\ddot{x}}{\dot{y}^2}$, and $rr\ddot{x} = a\dot{y}^2 + x\dot{y}^2$,

Fig. 18. $x\dot{y}^3$, assuming r constant. Therefore $rr\dot{x}\ddot{x} = a\dot{x}\dot{y}^2 + x\dot{x}\dot{y}^2$, and the fluent is $\frac{rr\dot{x}^2}{2} = a\dot{x}\dot{y}^2 + \frac{xx}{2}\dot{y}^2$

or $rr\dot{x}^2 = \overline{2ax + xx} \times \dot{y}^2$, whence $\dot{y} = \frac{r\dot{x}}{\sqrt{2ax+xx}}$

and the fluent is $y = 2rL \times \log. \frac{\sqrt{x} + \sqrt{2a + x}}{\sqrt{2a}}$,

or $y = rL \times \log. \frac{a+x+\sqrt{2ax+xx}}{a}$, which is to the ordinate of the catenary, as r to a. See Ex. 12. Prop. XIII. Fluxions. Hence assuming x successively 1, 2, 3, 4, &c. the correspondent values of y will be found, which may be put into a table.

Otherwise thus.

Since $rr\ddot{x} = a\dot{y}^2 + x\dot{y}^2$, assume

$x = Ay^2 + By^4 + Cy^6 + Dy^8$ &c.

then $\dot{x} = 2Ay + 4By^3 + 6Cy^5 + 8Dy^7$ &c.

and $\ddot{x} = 2A + 3.4By^2 + 5.6Cy^4 + 7.8Dy^6$ &c.

puting $\dot{y} = 1$.

Whence $rr\ddot{x} = 2rrA + 3.4rrBy^2 + 5.6rrCy^4 + 7.8rrDy^6$ &c.

$= a\dot{y}^2 + x\dot{y}^2 \} = a + Ay^2 + By^4 + Cy^6$ &c.

And equating the co-efficients $A = \frac{a}{2rr}$, $B = \frac{A}{3.4rr}$, $C = \frac{B}{5.6rr}$, $D = \frac{C}{7.8rr}$, &c. whence $x = \frac{a}{2} \times \frac{yy}{rr} + \frac{a}{2} \times \frac{1}{3.4} \times \frac{y^4}{r^4}$ &c. put $p = \frac{yy}{rr}$, then

$x = \frac{a}{1.2}p + \frac{a}{1.2.3.4}pp + \frac{a}{1...6}p^3 + \frac{a}{1...8}p^4$ &c.

Or

Or putting A, B, C, D, for the preceding terms, Fig. 18.

$x = \frac{a}{1.2} p + \frac{p}{3.4} A + \frac{p}{5.6} B + \frac{p}{7.8} C + \frac{p}{9.10}$ D &c.

To find r, put $x = ZC = 30$ feet, $y = AC = 30$ feet, $a = 3\frac{1}{2}$ feet, then $\frac{x}{a} = \frac{p}{1.2} + \frac{p}{3.4} A + \frac{p}{5.6} B + \frac{p}{7.8} C$ &c. $= \frac{30}{3\frac{1}{2}} = 8.57143$, and by reversion of series $p = 8.69774 = \frac{yy}{rr} = \frac{900}{rr}$, whence $rr = \frac{900}{p} = 103.4754$, which being known, put y successively equal to 0, 1, 2, 3, 4, &c. from which the several values of p will be found, by the equation $\frac{yy}{rr} = p$; and from these the respective values of x by the series above.

EXAM.

Let $y = 10$, then $p = \frac{100}{103.475} = .96642$, then

$$x = \frac{ap}{2} = 1.69130$$
$$+ \frac{Ap}{3.4} = .13621$$
$$+ \frac{Bp}{5.6} = .00439$$
$$+ \frac{Cp}{7.8} = .00007$$
$$1.83197 = x$$

Fig. 18. By this method I computed the following Table, and from this Table I made that which I publiſhed in my book of *Mechanics*. Ex. 106.

values of y.	values of p.	values of x.	y.	p.	x.
feet 1	0.00967	0.0169	16	2.47402	5.2991
2	.03867	0.0678	17	2.79294	6.1366
3	.08699	0.1533	18	3.13120	7.0673
4	.15463	0.2741	19	3.48877	8.1001
5	.24161	0.4314	20	3.86567	9.2451
6	.34792	0.6267	21	4.26190	10.5136
7	.47356	0.8619	22	4.67746	11.9174
8	.61853	1.1393	23	5.11234	13.4704
9	.78280	1.4616	24	5.56655	15.1875
10	.96642	1.8319	25	6.04010	17.0855
11	1.16938	2.2537	26	6.53298	19.1825
12	1.39165	2.7312	27	7.04518	21.4788
13	1.63326	3.2690	28	7.57670	24.0569
14	1.89420	3.8722	29	8.12755	26.8815
15	2.17444	4.5467	30	8.69774	30.0000

I alſo computed another Table upon the ſame principles, making a or $ZS = 7$ feet; and this Table is alſo publiſhed in the laſt edition of the *Mechanics*. By the ſame method other Tables may be computed for any other values of a. I ſhall here ſet down the values of p and rr for ſome other values of a.

If $a = 4$ feet, then $p = 8.007403$, and $rr = 112.396$;
$a = 5$, $p = 6.937512$, $rr = 129.7295$.
$a = 6$, $p = 6.139939$, $rr = 146.5813$.
$a = 7$, $p = 5.518200$, $rr = 163.0970$.
$a = 24$, $p = 2.104166$, $rr = 427.7229$.
$a = 28$, $p = 1.842124$, $rr = 488.5665$.

Cor.

Cor. 1. *When* $r = a$, *this curve becomes the tate-* Fig.
nary, whose latus rectum is a. 18.

For then they have every where the same ordinates, $aL \times \log. \frac{a+x+\sqrt{2ax+xx}}{a}$.

Cor. 2. *As* r : *to* $\sqrt{2ax+xx}$: : *rad. tan. angle* FAC.

For rad. : tan. FAC : : $\dot{y} : \dot{x} : : r : \sqrt{2ax+xx}$; where $x = ZC$, and the like for the angle at F, when $ZI = x$.

Cor. 3. *If* $ZI = x$, *the radius of curvature at* F *is* $= \frac{\overline{rr+2ax+xx}|^{\frac{3}{2}}}{r \times \overline{a+x}}$.

For this rad. $= \frac{\dot{x}^3}{\dot{y}\ddot{x}}$; and $\dot{y} = \frac{r\dot{x}}{\sqrt{2ax+xx}}$, whence $\dot{x} = \frac{\sqrt{2ax+xx}}{r}$ (putting $\dot{y} = 1$), and $\ddot{x} = \frac{a+x}{rr}$; therefore $\frac{\dot{x}^3}{\dot{y}\ddot{x}} = \frac{\dot{x}^3}{\dot{y}} \times \frac{rr}{a+x} = \frac{\overline{\dot{x}^2+\dot{y}^2}|^{\frac{3}{2}}}{\dot{y}} \times \frac{rr}{a+x} = \overline{\frac{2ax+xx}{rr}+1}|^{\frac{3}{2}} \times \frac{rr}{a+x} = \frac{\overline{rr+2ax+xx}|^{\frac{3}{2}}}{r \times \overline{a+x}}$.

Cor. 4. *Hence an arch of equilibration* AZR *may* 17.
be found of any given height and breadth CZ, AR; *and to have any degree of ascent* BED.

For instead of taking $AC = CZ$ as before, take them equal to any given numbers, and proceed as before. Thus if $x = 20$, and $y = 30$, $a = 5$. Then $\frac{20}{5}$ or $4 = \frac{p}{1.2} + \frac{p}{3.4}A + \frac{p}{5.6}B$ &c. and by reversion p is had, then $rr = \frac{yy}{p} = \frac{900}{p}$, and then

Fig. each value of y is found as before. And the point D is found by Prop. VII. for the aſcent BED.

Or having the curve conſtructed, where the height and half the breadth are equal, as in this Prop. a ſegment AZR may be cut off by trials, whoſe height to the breadth is in any given ratio, leſs than 1 to 2.

PROP. IX. PROB.

19. *Let* AFZR *be an arch ſupporting a fluid* APQR, *ſo that all the parts* F *of the arch ſhall remain in equilibrio*; gSQ *being a horizontal right line. To find the nature of the curve* ZF.

Let ZS $= a$, SI $= x$, FI $= y$, ZF $= z$. Then ſince the preſſure of the fluid at F, againſt the arch is as the depth Fg; and the ſtrength at F is as the curvature at F; therefore the height Fg is as the curvature at F, or reciprocally as the radius of curvature, and that is reciprocally as $\frac{\dot{z}\dot{x}}{-\ddot{y}}$, $\dot{z}$ being given; that is $\frac{-\ddot{y}}{\dot{z}\dot{x}}$ is as x, or $\frac{-rr\ddot{y}}{2} = \dot{z}x\dot{x}$ (aſſuming the given quantity $\frac{rr}{2}$); and taking the fluent, $-rr\dot{y} = xx\dot{z}$, and by correction $rr\dot{z} - rr\dot{y} = \overline{xx - aa} \times \dot{z}$, and $\overline{rr + aa - xx} \times \dot{z} = rr\dot{y}$, or (putting $rr + aa = bb$) $\overline{bb - xx}.\dot{z} = rr\dot{y}$, and $\overline{bb - xx}^2 \times \dot{z}^2 = \overline{bb - xx}^2 \times \dot{x}^2 + \dot{y}^2 = r^4\dot{y}^2$, which reduced, gives $\dot{y} = \frac{\overline{bb - xx} \times \dot{x}}{\sqrt{rr - b^4 + 2bbxx - x^4}}$, for the nature of the curve; and y may be found by infinite ſeries.

SCHOL.

SCHOL. Fig.

From hence, if a fluid could be made to ſtand upon an arch, the curvature at F being as the height Fg, muſt be greater at the bottom and leſs at the top of the arch. But if the fluid ſhould be congealed into ice or ſome ſolid body, the preſſure would not be the ſame; for the preſſure would then be directed downward, and not directly againſt the arch. 19.

PROP. X.

If AZFR *be an arch ſupporting the wall* APQR, *which wall conſiſts of perpendicular columns* Fg, *and that each of theſe ſtand upon the part of the curve* Fo; *and was only kept from ſliding down the arch by the next adjoining column. I ſay the curvature at* F, *muſt be as the height* Fg, *that all the parts may remain in equilibrio.* 20.

Let ZI $= x$, IF $= y$, ZF $= z$. Then ſince $\dot{z}$ is given, the weight of the column Fog will be $\dot{y} \times$ height or $\dot{y} \times$ Fg. Draw F*s* perp. to the curve, and *sr* perp. to IF or perp. to the horizon. Then *rs* will repreſent the weight of og, *r*F its preſſure againſt Fg, and F*s* the preſſure againſt the curve. Whence the weight $(\dot{y} \times Fg)$: preſſure againſt the curve : : *rs* : F*s* : : $\dot{y} : \dot{z}$. Therefore the preſſure againſt Fo $= \frac{\dot{z} \times \dot{y} \times Fg}{\dot{y}} = Fg \times \dot{z}$. But the curvature at F muſt be as that preſſure, that is as $Fg \times \dot{z}$; that is, the curvature muſt be as Fg.

Cor. *Hence the curve* ZF *is the ſame as that in the laſt Prop. For it comes to the ſame caſe as in that Prop.*

PROP.

Fig.

PROP. XI.

22. *If* ZFA *be an arch ſupporting the wall* ZSg*f*, *which is made of arch ſtones* GF*fg lying aſlope, ſo that all the joints* GF, *gf*, *may be perpendicular to the curve of the arch* ZF*f*. *To find the length* FG *of any arch ſtone, in any place; ſo that the whole arch may reſt in equilibrio.*

Let Z be the vertex of the inner curve of the arch, ZS the height of the wall; HZ the rad. of curvature at Z, OF the radius of curvature at F. Take $CB = \sqrt{HS^2 - HZ^2}$, and through B draw BE parallel to the horizontal line CA. And ſuppoſing F*f*, the baſe of the ſtone FG*gf*, to be a ſmall part of the arch. Draw the radii FO, *f*O; and CD, C*d* parallels thereto; then the angle DC*d* = angle GO*g*. *Note*, the triangle CBD may be taken more commodiouſly out of the figure.

The ſtone FG*gf* is kept in equilibrio by its weight and the preſſure of the two plains GF, *gf*. But three forces ſuſtaining one another are as three lines drawn perpendicular to their directions; therefore if D*d* repreſent the gravity of the ſtone, DC will be the preſſure againſt the joint GF, and *d*C the preſſure againſt the joint *gf*; and the ſame *d*C muſt alſo be the preſſure againſt the next ſtone, ſince action and re-action are equal. And for the ſame reaſon, DC muſt be the preſſure of the next ſtone towards Z, againſt the joint GF, to preſerve the equilibrium, &c.

The ſectors OG*g*, OF*f*, and CD*d*, as alſo the fluxions thereof, are as OG^2, OF^2, and CD^2. Whence $OG^2 - OF^2 : CD^2 :: $ area GF*fg* : triangle DC*d*. And in the vertex it is $HS^2 - HZ^2 : CB^2 ::$ area SP : triangle CB*b*. But by conſtruction $HS^2 - HZ^2 = CB^2$; therefore area SP = triangle CB*b*.

And

And consequently, to preserve the equilibrium, the area $FGgf$ = triangle CDd, and $OG^2 - OF^2 = CD^2$, and $OG^2 = OF^2 + CD^2$. Fig. 22. 18.

Therefore to construct the exterior curve, find the center of curvature O for any given point of the curve F, and draw FO, and CD parallel to it, and produce OF to G, and make $FG = \sqrt{OF^2 + CD^2} - OF$, and G is a point in the curve. And thus as many points may be found as you please, and the curve SGg drawn through all these points will be the top of the arch required. An arch thus constructed will remain in equilibrio, as long as the arch stones have liberty to slide down their inclined planes on which they lie.

EXAM. 1.

Let AZR be the arch of a circle; and let ZS be the height at the vertex Z, and make $CB = \sqrt{HS^2 - HZ^2}$, and draw BE parallel to AR. From the center C draw CF, and produce it to G, so that FG may be equal $\sqrt{CF^2 + CD^2} - CF$. Then G will be in the curve. If $CZ = 8$, $ZS = 1$, $\angle ZCF = 45°$; then $CD = 5.83$, $FG = 1.9$. 21.

EXAM. 2.

Let ZFA be an ellipsis, whose latus rectum is l. Take any point F, at which point suppose a tangent to the curve to be drawn, and draw FO perpendicular to the tangent, and whose length $FO = \frac{FI^2}{ll}$. And from O, draw OFG, Ofg; and let S be the top of the arch. Take $CB = \sqrt{HS^2 - HZ^2}$, and draw BE parallel to CA, and CD parallel to OF; and make $FG = \sqrt{OF^2 + CD^2} - OF$, and G will be a point in the curve. 22.

If $AC = 8$, $CZ = 6$, $ZS = 1$, and $ZH = 10.66$, $SH = 11.66$, whence $CB = 4.73$; then if the ordinate

Fig. 22. nate at F, perpendicular to ZC be = 5, then the ſubtangent to the axis ZC, is = 2.99; whence the angle BCD = 30 53, < FIA = 59 7, and FI = 5.46, and FO = $\left(\frac{4FI^3}{ll}\right)$ 8.06, where $l = 9$, the latus rectum of AC, and CD = 5.51, whence at laſt FG = 1.71.

EXAM. 3.

If ZFA be a parabola whoſe latus rectum is 8, CZ = 4, ZS = 1, CA = 4. Having taken any point F; by the help of the abſciſſa and ordinate, find the ſubtangent, and the angle the tangent makes with the ordinate, which is the ſame angle the perpendicular FI produced makes with the axis ZC; ſuppoſe the ſubtangent and ordinate to be equal, that angle will be 45 degrees, and BCD = 45 deg. let R = radius of curvature at F = $2\sqrt{32}$, rad. curvature at Z = 4, whence CB = 3, CD = $\sqrt{18}$. And FG = $\sqrt{RR + CD^2} - R$ = FG = .77.

EXAM. 4.

Let ZF be the catenary, and let the tangent at F make an angle of 45, with the axis, and FI will make the ſame angle, and in this caſe ZF = a, let $a = 6$, ZS = 1, radius of curvature in Z = $a = 6$, whence CB = $\sqrt{13}$, and angle BCD alſo = 45, therefore CD = $\sqrt{26}$. And radius of curvature in F = $a + \frac{zz}{a} = 2a = 12$, becauſe $z = a$. Whence FG $\sqrt{144+26} - 12 = 1.04$.

EXAM. 5.

Let ZF be a cycloid, the diameter of the generating circle CZ = 6, ZS = 1, rad. curvature in Z = 2CZ = 12, and ſuppoſe the ordinate drawn to F, to paſs through the center of the generating circle;

cle; then the tangent at F, and also the perp. GI will cut the axis ZC at an angle of 45; in which case the radius of curvature at F $= \sqrt{2CZ^2} = \sqrt{72} = 8.48$, and CB $= 5$, CD $= \sqrt{50}$. Whence FG $= \sqrt{122} - 8.48 = 2.56$. This is the best curve tried yet for the figure of an arch of this sort. Fig. 22.

SCHOL.

None of the three cases mentioned in the three last propositions can hold good in the construction of arches. For the wall is not in the nature of a fluid (Fig. 19.) nor do the columns endeavour to slide down in direction oF (Fig. 20.) or in direction *gf* (Fig. 21.) For when the parts are cemented together by the mortar, they will all act perpendicular to the horizon, and are each fixed to the parts of the arch, as supposed in Prop. III.

What has been demonstrated in Prop. VII. in regard to the arch AZR (Fig. 15.) holds true in these last arches also, only taking the joints instead of perpendicular lines. That is, if ZS, FG, *fg*, and all the other joints, be divided in any given ratio, and a curve line drawn through all the points, for the top of the arch; the arch will still remain in equilibrio. Or if the joints be increased in any given ratio, the line drawn through all the points will be the top of the arch, which will still be in equilibrio. See also Prob. 58. Art. 18.

PROP. XII.

If SM *be a line parallel to the horizon, and* ZQ *a curve of such a nature, that drawing* LQM, *and* CN *parallel to it; so that* LQ *be the radius of curvature at* Q, *and* $\overline{2LQ + QM} \times QM$ *be always equal to* CN^2; *then all the parts of the arch* ZSMQ *will remain in equilibrio.*

For

Fig. 22. For by Prop. XI. if $CB = \sqrt{HS^2 - HZ^2}$, and $LM^2 - LQ^2 = CN^2$, then the arch ZQ will ſtand in equilibrio. But $LM = LQ + QM$, and $LM^2 - LQ^2 = LQ^2 + 2LQ \times QM + QM^2 - LQ^2 = CN^2$, that is, $2LQ \times QM + QM^2 = CN^2$, therefore all the parts of the arch bounded by the right line SM and the curve ZQ will remain in equilibrio.

But the calculation of ſuch a curve is ſo very intricate, that it can be of no manner of uſe in the building of bridges.

PROP. XIII.

23. *If any curve* AFZ *be made to revolve round its axis* ZC, *and form thereby a concave ſurface or vault; and if a wall* SZFg *be erected thereon, which is to be ſupported by that ſurface, like an arch. Then the height of that wall at any point* F, *ſo that all the parts may ſtand in equilibrio, that is* Fg, *will be as* $\frac{FT^3}{FI^4 \times R}$, *where* FT *is a tangent at* F, FI *an ordinate, and* R *the radius of curvature in* F.

Let ZAC, Z*a*C be two planes, comprehending the ſmall part of the vault Z*Aa*; then the diſtance F*f* at any place F will be as the ordinate FI; and the incumbent weight on the line F*f* will be $Fg \times Ff$, or as $Fg \times FI$.

Let $ZI = x$, $FI = y$, $ZF = z$, and let $\dot{z}$ be a given part of the curve ZF; and $\dot{y}$ a ſmall part of the ordinate FI. Then $\dot{y} = \frac{\dot{z} \times FI}{FT}$, and $\dot{y}$ is as $\frac{FI}{FT}$, becauſe $\dot{z}$ is given. Then the incumbent weight on the ſmall part of the ſurface at F is as $Fg \times FI$

Fig. 23.

$FI \times \dot{y}$ or as $Fg \times \frac{FI^2}{FT}$. By Prop. II. the preſſure at any point F is as $\frac{C=\text{curv.}}{S.ITF^2} = C \times \frac{FT^2}{FI^2}$. Whence $Fg \times \frac{FI^2}{FT}$ is as $C \times \frac{FT^2}{FI^2}$ or as $\frac{FT^2}{FI^2 \times R}$, and Fg as $\frac{FT^3}{FI^4 \times R}$.

Cor. 1. *Hence* Fg *is as* $\frac{-\dot{x}\ddot{y}}{\dot{y}^3}$, *putting* $\dot{x}$ *invariable.*

For $\frac{FT}{FI} = \frac{\dot{z}}{\dot{y}}$, and $R = \frac{\dot{z}^3}{-\dot{x}\ddot{y}}$; therefore Fg being as $\frac{FT^3}{FI^4 \times R}$ is as $\frac{\dot{z}^3}{\dot{y}^3 y R}$ or as $\frac{\dot{z}^3}{\dot{y}\dot{y}^3} \times \frac{-\dot{x}\ddot{y}}{\dot{z}^3} = \frac{-\dot{x}\ddot{y}}{\dot{y}\dot{y}^3}$, as is alſo ſhewn in my book of Fluxions.

Or thus, $R = \frac{\dot{z}^3}{\dot{y}\ddot{x}}$ when $\dot{y}$ is given, whence Fg is as $\frac{\dot{z}^3}{\dot{y}\dot{y}^3} \times \frac{\dot{y}\ddot{x}}{\dot{z}^3} = \frac{\ddot{x}}{\dot{y}\dot{y}^2}$.

Cor. 2. *Hence no curve can produce the figure of a true vault, except the radius of curvature in the vertex be infinite.*

For in the vertex, FT = FI, and Fg in the vertex is $\frac{FI^3}{FI^4 \times R} = \frac{1}{FI \times R} = \text{infinity}$, when R is finite, and FI = o, therefore to make Fg a real quantity, R muſt be infinite; otherwiſe ZS will be infinite.

EXAM. I.

Let FZR be the common parabola, $rx = yy$, then 24. $IT = 2x$, and $FT = \sqrt{yy + 4xx} = \sqrt{rx + 4xx}$, and

R =

Fig. $R = \frac{\overline{rr+4yy}|^{\frac{3}{2}}}{2rr}$. Therefore Fg is as $\frac{\overline{rx+4xx}|^{\frac{3}{2}}}{y^4R}$ or as $\frac{\overline{rx+4xx}|^{\frac{3}{2}}}{rrxx} \times \frac{2rr}{\overline{rr+4rx}|^{\frac{3}{2}}}$ that is, Fg is as $\frac{x^{\frac{3}{2}}}{r^{\frac{3}{2}}xx}$ or as $\frac{1}{\sqrt{x}}$, or Fg is as $\frac{1}{y}$; and therefore at the vertex Fg is infinite. And so it will in the circle.

EXAM. 2.

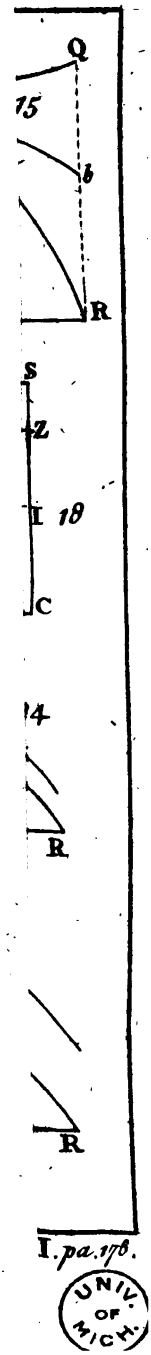
25. Let AFZR be a cubic parabola, whose equation is $x = y^3$. Here $IT = 3x$, and $FT = \sqrt{yy+9xx} = \sqrt{yy+9y^6}$. And $R = \frac{\overline{1+9y^2}|^{\frac{3}{2}}}{6y}$, whence Fg is as $\frac{FT^3}{FI^4R}$ or as $\frac{\overline{yy+9y^6}|^{\frac{3}{2}}}{y^4} \times \frac{6y}{\overline{1+9y^2}|^{\frac{3}{2}}} = \frac{6y \times y^3}{y^4}$ or as 6 a given quantity. Therefore Fg is every the same, and the curve Sg is also a cubic parabola, the same with the former, but in a higher position.

After the same manner if AFZ be a biquadratic parabola, where $x = y^4$, there will come out Fg as y; and therefore the height at the vertex is 0.

PROP. XIV. PROB.

26. *To investigate and construct the curve or arch* AFZ, *by whose revolution a vault* AZR *is formed, upon which the wall or building* APQR *being erected, and being bounded by the horizontal line* PQ, *the height* ZS *being given; all the parts may remain in equilibrio.*

Draw ZG parallel to QSP, and let $SZ = a$, $ZI = x$, $FI = y$, $AC = b$, $ZC = b$. Then $Fg = a+x$, but (by Cor. 1. Prop. XIII.) Fg is as $\frac{\ddot{x}}{\dot{y}\dot{y}^2}$; whence $\frac{\ddot{x}}{\dot{y}\dot{y}^2}$



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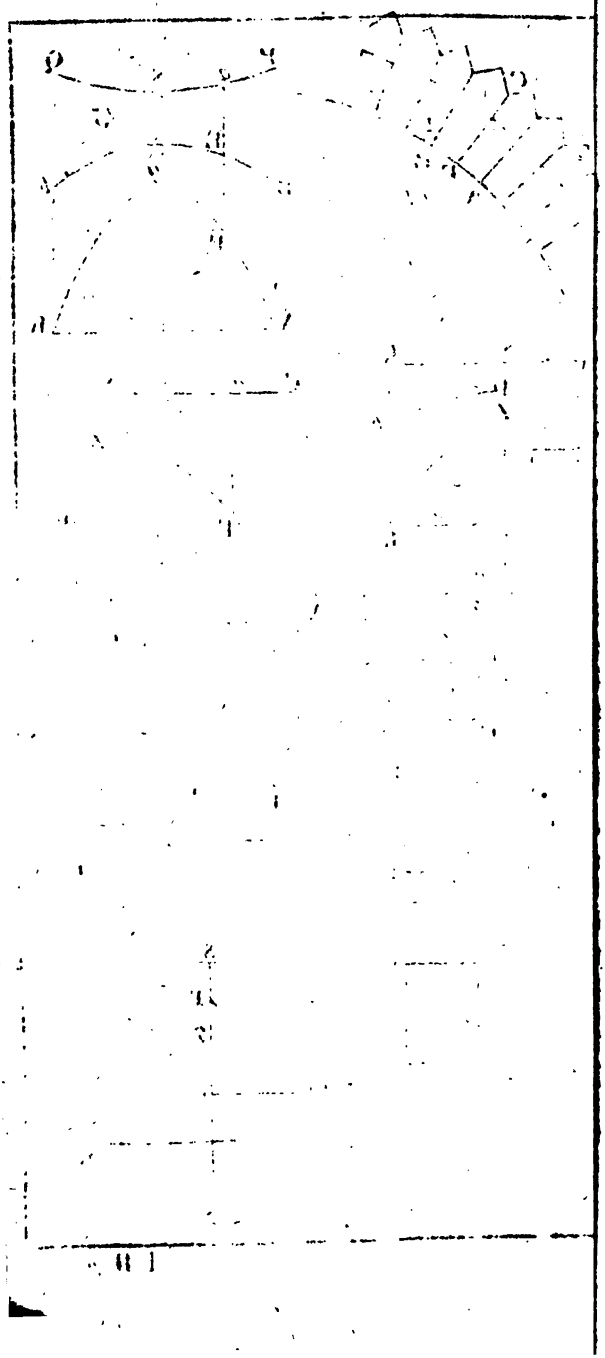


Fig. 26.

$\frac{\ddot{x}}{\dot{y}\dot{y}^2}$ is as $a+x$; and taking the conſtant quantity r^3, $r^3\ddot{x} = a\dot{y}\dot{y}^2 + x\dot{y}\dot{y}^2$, $\dot{y}$ being $= 1$.

Finding the indices (by Rule III. Prop. X. Sect. 1. Fluxions) to be 3, 6, 9, 12, &c.

Aſſume $x =$	$Ay^3 +$	$By^6 +$	$Cy^9 +$	Dy^{12} &c.	
then $\dot{x} =$	$3Ay^2 +$	$6By^5 +$	$9Cy^8 +$	$12Dy^{11}$ &c.	
and $\ddot{x} =$	$2.3Ay +$	$5.6By^4 +$	$8.9Cy^7 +$	$11.12Dy^{10}$ &c.	
then $r\dot{r}\ddot{x} =$	$2.3r^2Ay +$	$5.6r^2By^4 +$	$8.9r^2Cy^7 +$	$11.12r^2Dy^{10}$ &c.	
$= ay + xy$	$= ay +$	$Ay^4 +$	$By^7 +$	Cy^{10} &c.	

And equating the co-efficients $A = \frac{a}{2.3r^2}$, $B = \frac{A}{5.6r^2}$, $C = \frac{B}{8.9r^2}$, $D = \frac{C}{11.12r^2}$, &c. put $p = \frac{y^2}{r^2}$.

Whence $x = \frac{ay^3}{2.3r^2} + \frac{ay^6}{2.3.5.6r^6} + \frac{ay^9}{2.3.5.6.8.9r^9}$ &c. $= \frac{a}{2.3}p^3 + \frac{a}{2.3.5.6}p^6 + \frac{a}{2.3.5.6.8.9}p^9$ &c.

Or putting A, B, C, D, &c. for the preceding terms,

$$x = \frac{a}{2.3}p + \frac{p}{5.6}A + \frac{p}{8.9}B + \frac{p}{11.12}C + \frac{p}{14.15}D \text{ &c.}$$

Let SZ or $a = 1$, $x = b = 10$, $y = b = 15$. Then to find r, we have $\frac{b}{a} = \frac{p}{6} + \frac{p}{30}A + \frac{p}{72}B + \frac{p}{132}C$ &c. $= 10$. And by reverſion of ſeries $p = 26.37 = \frac{b^2}{r^2}$, and $r^2 = \frac{b^2}{26.37} = \frac{3375}{26.37} = 128$.

And r^2 being known; put $y = 0, 1, 2, 3, 4$, &c. from which find the ſeveral values of p by the equation $\frac{y^2}{r^2} = p$, and from theſe the correſpondent values of x by the foregoing ſeries.

Fig. 26.

Eaſier thus.

Inſtead of aſſuming ſome value for a, aſſume the value of r^3, then the equation $\frac{b^3}{r^3} = p$, gives p; which being known, the ſeries $\frac{p}{6} + \frac{p}{30} A + \frac{p}{72} B$ &c. gives $\frac{b}{a}$; whence a will be known. But if a is not of a commodious length, aſſume r^3 anew, till a be of a proper magnitude.

For example, take $r^3 = 130$, then $p = \frac{3375}{130} = 25.9615$, in this caſe. Then $\frac{b}{a} = \frac{1}{6} p + \frac{A}{30} p + \frac{B}{72} p + \frac{C}{132} p + \frac{D}{210} p$ &c. $= 9.72285$, and $a = \frac{10}{9.72285} = 1.02851$. Now put $p = \frac{y^3}{r^3}$, and aſſume $y = 0, 1, 2, 3, 4$, &c. ſucceſſively, then ſo many values of p will be had, and then the correſpondent values of x, by the ſeries $x = \frac{a}{2.3} p + \frac{p}{5.6} A + \frac{p}{8.9} B + \frac{p}{11.12} C$ &c. And theſe being all computed will be as in the following Table.

It

Fig.

values of y.	values of p.	values of x.	values of y.	values of p.	values of x.
1	.00769	.0013	9	5.60770	1.1555
2	.06154	.0106	10	7.69231	1.6950
3	.20769	.0358	11	10.23846	2.4461
4	.49231	.0858	12	13.29231	3.4945
5	.96154	.1702	13	16.90000	4.9652
6	1.66154	.3009	14	21.10769	7.0425
7	2.63846	.4935	15	25.96154	10.0000
8	3.93846	.7687			

It is eaſy to conſtruct this curve by the Table, ſetting off ZD or y by the firſt column; and DF or x, by the third; and drawing a curve through all the points. The curve is very flat at the top, and at the bottom riſes at an angle of about 71 $\frac{1}{2}$ degrees.

Fig.

ART. VI.

The Preceſſion of the Equinoxes.

IN Prob. XXVI. of my book of Fluxions, I have given a calculation of the Preceſſion of the Equinoxes, *ab origine*, that is, depending upon nothing but the laws of motion. What I give here is deduced primarily from the motion of the moon's nodes, which makes the calculations vaſtly ſhorter.

PROP. I. PROB.

27. *Suppoſing the denſity of the earth the ſame throughout, to find the preceſſion of the equinoxes, by the ſun's force.*

Let QAR be the earth, CA its radius, and put
a = earth's radius CA,
b = 72706″, motion of the moon's nodes in a year,
n = 1436′, a ſyderial day,
d = 40217′ the moon's periodic time,
e = 52441 the matter in the earth,
l = 459 the matter in the cruſt,
x = col. 5°, and c = cof. 23° ½, f = generating force.

By Cor. 3. Prop. 28. *Centripetal Forces*, $d : n :: b : \frac{bn}{d}$ the yearly motion of the nodes of a moon, revolving in the time n, or of an equinoctial ring.

Now by Prop. 59. *Mechanics*, the ſame angular motions will be produced, if the bodies of the ring, and of the earth, be placed in their centers of gyration,

gyration, or at the distances $a\sqrt{\frac{1}{2}}$ and $a\sqrt{\frac{2}{3}}$; therefore (by Cor. 2. Prop. 56. ib.) it will be as $\frac{f}{l\times\frac{1}{2}aa}$: $\frac{f}{l\times\frac{1}{2}aa+e\times\frac{2}{3}aa}$ :: so angular motion of the equinoctial ring $\frac{bn}{d}$: $\frac{\frac{1}{2}l\times bn}{d\times\frac{1}{2}l+\frac{2}{3}e}$ = motion of the nodes of the whole earth; that is, when the force (f) acts altogether upon the matter l in the equinoctial ring. But the force upon the crust or exterior part, diffused over the whole surface of the sphere, is but $\frac{2}{3}$ of the equinoctial force. Therefore $\frac{\frac{2}{3}\times\frac{1}{2}lbn}{d\times\frac{1}{2}l+\frac{2}{3}e}$ or $\frac{lbn}{d\times\frac{3}{4}l+2e}$ = motion of the earth's nodes, supposing the equinoctial elevated above the ecliptic, no more than the moon's orbit. But the force (and consequently the motion) will be less in proportion to the cosines of elevation; therefore $x : c$ :: motion of e : $\frac{clbn}{xdw}$ = the true motion of the equinoxes, by the sun's force; putting $\frac{3}{4}l + 2e = w$ = 106030.

Fig. 27.

Operation.

log. c =	9.962398
l =	2.661813
b =	4.861570
n =	3.157154
	20.642935
x =	9.998344
d =	4.604409
w =	5.025429
	19.628182
log.	1.014753

Motion of the equinoxes 10″.345, by the sun.

Fig. 27.

PROP. II.

Suppoſe the denſity of the earth to be reciprocally as the n^{th} power of the diſtance from the center; then the angular motion of the nodes in the uniform ſphere: is to the angular motion of the nodes in this variable ſphere :: as 5 : to $5-n$; ſuppoſing both of the ſame denſity at the ſurface.

Let CA $= r$, CB or CO $= x$, $c = 3.1416$, $d =$ denſity at A at the ſurface. Then $\frac{r^n d}{x^n}$ will be the denſity at B; and the ſpherical ſurface at B $= 4cxx$; and by the variable denſity, the matter in the orb at B $= 4cx^2\dot{x} \times \frac{r^n}{x^n} d = 4cdr^n x^{2-n}\dot{x}$. But (by Ex. 5. Prob. 23, *Fluxions*), the diſtance of the center of gyration of the orb BO, is $= x\sqrt{\frac{2}{3}} =$ Cp, and of the ſphere $r\sqrt{\frac{2}{3}}$. Therefore (by Cor. 2. Prop. 56. *Mechanics*) the force remaining the ſame, the angular motion of the nodes in the uniform ſphere: to that in the invariable ſphere :: $\frac{1}{\frac{2}{3}rr \times \text{ſphere}}$: $\frac{1}{\text{ſum of all } cp^2 \times 4cdr^n x^{2-n}\dot{x}}$. But the ſphere $= \frac{4}{3}cr^3 d$. Alſo the ſum of all the C$p^2 \times 4cdr^n x^{2-n}\dot{x}$, or of all the $\frac{2}{3}xx \times 4cdr^n x^{2-n}\dot{x}$, or of all $\frac{8}{3}cdr^n x^{4-n}\dot{x} = \frac{8}{3}cdr^n \times$ fluent of $x^{4-n}\dot{x} = \frac{8}{3}cdr^n \times \frac{x^{5-n}}{5-n}$. But when $n = 5$, $\frac{8}{3}cdr^n \times$ fl. $\frac{\dot{x}}{x} = \frac{8}{3}cdr^5 \times$ log. x. Whence the angular motion in the uniform ſphere: to the angular motion in the variable ſphere :: $\frac{1}{\frac{8}{15}cdr^5}$:

$\frac{1}{\frac{8}{15}cdr^n \times \frac{x^{5-n}}{5-n}}$, or when $x = r$, as $\frac{8}{15}cdr^n \times \frac{r^{5-n}}{5-n} : \frac{8}{15}cdr^5$, or as $\frac{15r^5}{5-n} : 3r^5$, or as 5 to $5-n$. Fig. 27.

Cor. 1. *When* $n = 1$, *or the denſity reciprotally as the diſtance from the center; the angular motions in the uniform and variable ſpheres, will be as* 5 *to* 4, *for the motion of the equinoxes.*

Cor. 2. *If* $n = 5$, *the motion of the equinoxes will be nothing.*

For the motion of the uniform ſphere is as log. x; but when $x = 0$, log. x = infinity, and the motion as log. x — log. 0 = log. $\frac{x}{0}$ = infinity. And therefore the motion of the variable ſphere, is 0 in reſpect to it.

Cor. 3. *If n is greater than* 5, *the motion is negative or backward.*

Cor. 4. *If n is negative, or which is the ſame thing, if the denſity be as* x^n; *then the motion of the equinoxes in the uniform and variable ſpheres, will be as* 5 *to* $5 + n$.

PROP. III.

Suppoſe the denſity at the center C *greater than that at the ſurface* A, *in any ratio as n to* 1. *The motion of the equinoxes in the uniform ſphere, is to that in the variable ſphere; as* $n + 5$ *to* 6; *ſuppoſing them both of the ſame denſity at the ſurface; and in the latter to increaſe uniformly from* A *to* C. 28.

Let AD $= d$ the denſity at the ſurface A, CG $= nd$ the denſity at the center, BC $= x$; draw BH

Fig. 28. parallel to CG or AD, then BH will be the denſity at B. Draw DF parallel to AC, and FG $= \overline{n-1}.d$. By ſimilar triangles DF : FG :: DE : EH, that is $r : \overline{n-1}.d :: r-x : EH = \frac{r-x}{r} \times \overline{n-1}.d$; and $BH = \frac{r-x}{r} \times \overline{n-1}.d + d =$ denſity at B=D, by ſubſtitution. Then $4cx^2\dot{x} \times D =$ matter in the orb at B. Then reaſoning as before, motion in the uniform ſphere, to the motion in the variable one; as $\frac{1}{\frac{8}{15}cdr^5}$ to $\frac{1}{\text{ſum of all } \frac{8}{3}x\dot{x} \times 4cx^4\dot{x} \times D}$; or as ſum of all the $\frac{8}{3}cx^4\dot{x} \times D$ to $\frac{8}{15}cdr^5$; but the ſum of all $\frac{8}{3}cx^4\dot{x} \times D =$ ſum of $\frac{8}{3}cx^4\dot{x}D =$ fluent of $\frac{8}{3}cx^4\dot{x} \times$ by $nd - \frac{dx}{r} \times \overline{n-1}$: $=$ fl. $\frac{8}{15}cndx^4\dot{x} - \frac{8cd}{3r}x^5\dot{x} \times \overline{n-1} = \frac{8}{15}cndx^5 - \frac{8cd.\overline{n-1}}{18r} \times x^6$. Therefore the motions in the uniform and variable ſphere, are as $\frac{8}{15}cndx^5 - \frac{8cd \times \overline{n-1}}{18r}x^6$ to $\frac{8}{15}cdr^5$. And when $x = r$, theſe motions are as : $\frac{8}{15}cnd - \frac{8}{18}cd.\overline{n-1} : \times r^5$ to $\frac{8}{15}cdr^5$, or as $\frac{8}{15}n - \frac{8}{18}n + \frac{8}{18} : \frac{8}{15}$, or as : $\frac{1}{15} - \frac{1}{18} \times n + \frac{1}{18} : \frac{1}{15}$, that is, as $n + 5$ to 6.

Cor. 1. *If* $n = 0$, *or the denſity be* 0 *at the center; the motion of the equinoctial points, in the uniform and variable ſpheres; will be as* 5 *to* 6.

Cor. 2. *If* $n = 1$, *theſe motions will be equal.* For then they are both uniform.

PROP.

Fig. 29.

PROP. IV. PROB.

Let AR *be a ſphere ſo formed, that the decreaſe of denſity, in going from the center to the circumference, as at* B, *may be as* CB^m, *that is, as any power m of the diſtance from the center, to find the motion of the equinoctial points.*

Let $AD = d$, be the denſity at A, $CG = b = nd$, be the denſity at C; and let DHG be the curve of denſity. Draw BEHO parallel to CG; and let CB or $GO = x$, $FG = a$, CA or $GK = r$. Then by nature of the curve GH, GK^m or CA^m : FG or KD : : CB^m or GO^m : OH; that is, $r^m : a :: x^m : OH = \frac{x^m}{r^m} a$, the decreaſe of denſity at B; therefore $b - \frac{x^m}{r^m} a$ = denſity at B; and $4cx^2\dot{x} \times \overline{b - \frac{x^m}{r^m} a}$ = matter in the orb at B. And the ſum of all the $\frac{2}{3}xx \times 4cx^2\dot{x} \times \overline{b - \frac{x^m}{r^m} a}$, or the fluent of $\frac{8}{3}cx^4\dot{x} \times \overline{b - \frac{x^m}{r^m} a} = \frac{8}{3 \cdot 5} cbx^5 - \frac{8cax^{m+5}}{3r^m \times \overline{m+5}}$. Whence the motions in the uniform and variable ſpheres are as $\frac{8}{3 \cdot 5} cbx^5 - \frac{8cax^{m+5}}{3r^m \times \overline{m+5}}$ and $\frac{8}{15}cdr^5$; and (when $x = r$) as $\frac{2}{5} br^5 - \frac{ar^{m+5}}{r^m \times \overline{m+5}}$ to $\frac{2}{5} dr^5$, or as $b - \frac{5a}{m+5}$ to d, that is as $nd - \frac{5}{m+5} \times \overline{n-1}.d$ to d, whence the motions in the uniform and variable ſpheres are as $n - \frac{5 \times \overline{n-1}}{m+5}$ to 1, or as $mn + 5$ to $m + 5$.

Cor.

Fig. 29. Cor. *If* $m = 1$, *the motions are as* $n + 5$ *to* 6, *agreeable to Prop.* III.

PROP. V. PROB.

27. *Suppose the denſity of the earth at any place* B, *to be as* $\frac{1}{CB^n}$; *going upwards from* B *to* A; *but that it is hollow below the ſurface* BFD; *to find the motion of the equinoctial points in this hollow ſphere; having that of a ſimilar hollow ſphere of uniform denſity, both having the ſame denſity at* A.

Let CA $= r$, $d =$ denſity at A, CB $= x$, proceeding as in Prop. II. The ſurface at B $= 4cxx$, and the matter in the orb $= 4cx^2\dot{x} \times d$ for the uniform ſphere, and for the variable ſphere $= 4cx^2\dot{x} \times \frac{r^n}{x^n} d$ or $4cdr^n x^{2-n}\dot{x}$; theſe multiplied by $\frac{2}{3}xx$ (the ſquare of the diſtance of the center of gyration), will be $\frac{8}{3}cdx^4\dot{x}$ for the uniform ſphere, and $\frac{8}{3}cdr^n x^{4-n}\dot{x}$ for the other. Therefore the motion in the uniform ſphere : motion in the variable one : : as $\frac{1}{\text{ſum} : \frac{8}{3}cdx^4\dot{x}}$: $\frac{1}{\text{ſum} : \frac{8}{3}cdr^n x^{4-n}\dot{x}}$: : fluent of $\frac{8}{3}cd \times r^n x^{4-n}\dot{x}$: fluent of $\frac{8}{3}cdx^4\dot{x}$: : fluent of $r^n x^{4-n}\dot{x}$: fluent of $x^4\dot{x}$: : $\frac{r^n x^{5-n}}{5-n}$: $\frac{x^5}{5}$, for the hollow parts BOFD. Put $x = r$, and the fluents are $\frac{r^5}{5-n}$ and $\frac{r^5}{5}$, for the whole ſpheres. Subtract the reſpective parts from the wholes, and the motion of the equinoxes, for the uniform and variable hollow ſpheres, will be as $\frac{r^5 - r^n x^{5-n}}{5-n}$ to $\frac{r^5 - x^5}{5}$.

Cor.

Fig. 27.

Cor. *The motion in the whole uniform ſphere, is to the motion in the variable hollow ſphere; as* $\frac{5}{r^5}$ *to* $\frac{5-n}{r^5-r^n x^{5-n}}$.

For the motions of the uniform and variable ſpheres are as $\frac{1}{\text{fl. } x^4\dot{x}}$ and $\frac{1}{\text{fl. } r^n x^{4-n}\dot{x}}$, or as $\frac{1}{\frac{1}{5}x^5}$ and $\frac{1}{\frac{1}{5-n} \times r^n x^{5-n}}$, or as $\frac{5}{x^5}$ and $\frac{5-n}{r^n x^{5-n}}$; and the mot. in the whole uniform ſphere, to the motion in the variable hollow ſphere; as $\frac{5}{r^5}$ to $\frac{5-n}{r^5-r^n x^{5-n}}$.

What is done here with regard to the ſecond propoſition; may as eaſily be done upon the ſuppoſitions laid down in the 3d and 4th propoſitions; and therefore I ſhall not trouble the reader with them. I ſhall only give one example of this Prop. and its Corol.

Let $n=1$, $x=\frac{1}{2}r=$CB. Then the mot. in the uniform and variable hollow ſpheres ABFDQ, are as $\frac{r^5-r\times\frac{r^4}{16}}{4}$ to $\frac{r^5-\frac{1}{32}r^5}{5}$ or as $\frac{15}{4\times16}$ to $\frac{31}{5\times32}$ that is, as 75 to 62.

And the motion in the whole uniform ſphere, to that in the variable hollow ſphere, is as $\frac{5}{r^5}$ to $\frac{4}{r^5-r\times\frac{r^4}{16}}$, or as 5 to $\frac{4\times16}{15}$ or as 75 to 64.

Hence if the motion of the equinoxes be known for a earth of uniform denſity, the motion of a earth formed according to any law of denſity will alſo be known, by the foregoing propoſitions.

And

Fig. And thus I have compleated that difficult problem of finding the preceſſion of the equinoxes by the ſun's force, upon any ſuppoſition of denſity.

SCHOL.

The moon's diſturbing force is not preciſely known, and we have not ſufficient *data* to determine it. Let P = earth's periodical time, or a year, and p = earth's periodical time about the moon at reſt; then the moon's diſturbing force = $\frac{PP}{pp}$ × ſun's diſturbing force. But as p is unknown; we muſt judge as well as we can from the foregoing propoſitions.

Preceſſion of the equinoxes by obſervation	50″.3
in a earth of uniform denſity, by the ſun	10.34
by Prop. 2. Cor. 1. $n = 1$.	8.27
by Prop. 3. $n = 2$.	8.86

Hence this may be concluded in general, whatever be the law of denſity, if it be greater within the earth, the motion by the ſun's force will be leſs; and conſequently the moon's force ſo much greater. Neither is there any way to make the moon's force leſs, but by making the earth hollow, or elſe rarer within, which is not probable. Hence we cannot well ſuppoſe, that the preceſſion by the ſun is greater than 10.34; nor conſequently that by the moon to be leſs than (50.3 — 10.34) 39.96; then $\frac{39.96}{10.34}$ = 3.86 for the moon's force, ſuppoſing the ſun's force 1. But by the foregoing Props. the preceſſion by the ſun muſt probably be leſs, and therefore its force leſs, and therefore the moon's bigger than is here ſet down.

ART.

Fig.

ART. VII.

The Conſtruction of Logarithms.

I Deſign not here to treat of the beſt and moſt expeditious methods of making logarithms, I have already done that in my book of Fluxions. I ſhall here only ſhew how this may be done by the indices or roots of numbers; by ſuch methods the Tables of logarithms were with great labour conſtructed at firſt. And though theſe methods are very tedious, and require a great deal of time and pains to perform, yet they ſhew the nature of logarithms, and their relation to the numbers themſelves in the cleareſt and plaineſt manner.

PROB I.

To make logarithms by the indices of the roots of 10.

This method depends on aſſuming ſome number as the logarithm of 10, and the ſeveral multiples thereof, for the logarithms of the correſponding powers of 10; and then finding the logarithms or indices for the intermediate powers. According to this method, ſuppoſe log. $1=0$, $10^0=1$, $10^1=10$, $10^2=100$, $10^3=1000$, &c. where 0, 1, 2, 3, &c. are the aſſumed logarithms of 1, 10, 100, 1000; then to find the logarithms of the intermediate numbers, ſuppoſe an immenſe ſeries of geometrical proportionals from 1 to 10, &c. Then all numbers may be ſuppoſed to be contained in this ſeries, and every number may be ſome power (or root) of 10; and the index expreſſing this power or root, will be the logarithm of that number; and the buſineſs

Fig. fineſs is to find that index or logarithm. This ſeries may be denoted thus, $1, x, x^2, x^3, x^4, x^5, x^6$, &c. .... to $x^{10000000} = 10$. And therefore to find the logarithm of any number is to find its place in the ſeries. And this may be done, by help of the following Table of the ſquare roots of 10. This Table is made by continually extracting the ſquare roots of 10, till the indices of x all vaniſh, as follows.

Square roots of 10.		powers of x.
	$=$	$x^{10000000}$
3.1622776	$=$	$x^{5000000}$
1.7782794	$=$	$x^{2500000}$
1.3335214	$=$	$x^{1250000}$
1.1547819	$=$	x^{625000}
1.0746078	$=$	x^{312500}
1.0366329	$=$	x^{156250}
1.0181517	$=$	x^{78125}
1.0090350	$=$	x^{39062}
1.0045073	$=$	x^{19531}
1.0022511	$=$	x^{9765}
1.0011249	$=$	x^{4883}
1.0005623	$=$	x^{2441}
1.0002811	$=$	x^{1221}
1.0001405	$=$	x^{610}
1.0000703	$=$	x^{305}
1.0000351	$=$	x^{152}
1.0000175	$=$	x^{76}
1.0000087	$=$	x^{38}
1.0000044	$=$	x^{19}
1.0000022	$=$	x^{9}
1.0000011	$=$	x^{5}
1.0000005	$=$	x^{2}
1.0000003	$=$	x^{1}
1.0000001	$=$	x^{1}

In this Table it may be obſerved, that any index of x, divided by 10000000, is the logarithm of

of its correſpondent number. Thus 1 is the log. of 10, and .5 is the log. of 3.1622776; and .25 is the log. 1.7782794; and ſo on through the Table. And ſince the addition of theſe indices or logarithms, anſwers to the multiplication of the correſpondent numbers; therefore to find the log. of any number by this Table, we muſt ſeek out ſuch numbers in the Table, that being multiplied together (or perhaps divided), they will produce the given number; ſo the indices being added (or perhaps ſubtracted) will give the log. of that number. Therefore to find the logarithm of any number, Fig.

Proceed thus. Take the neareſt number thereto in the Table (on the left hand), which ſet down with the power or index of x againſt it. If it be too little, multiply by ſome following number in the Table, found by trials, ſuch that the product may come the neareſt to the given number; but if it was too great, divide by ſuch a number. Set down that number and the index of x againſt it; then below, ſet down the product (or quotient) of the numbers, and againſt it, the ſum (or difference) of the indices.

Go on the ſame way, by continually multiplying (or dividing) by leſs and leſs numbers, and adding (or ſubtracting) their indices, as you proceed, and thus you will continually approximate to the number and to its logarithm. So that at laſt when you have obtained the number, and its index; divide the index by 10000000, and you have the logarithm.

Note 1. To find the next multiplier expeditiouſly (without too many trials); divide the given number, by the laſt number, ſo far as to get a figure or two in decimals; then the quotient will point out the multiplier in the Table. Or divide, to get a diviſor.

Note

Fig. *Note* 2. You may always uſe multiplication if you pleaſe, provided you never make a product to exceed the number given; but the work will go on ſlower.

EXAM.

To find the logarithm of 2.

	Num.	Indexes.
	1.7782794	2500000
×	1.1547819	625000
pr.	2.0535247	3125000
÷	1.0366329	156250
qu.	1.9809567	2968750
×	1.0090350	39062
pr.	1.9988546	3007812
×	1.0005623	2441
pr.	1.9999785	3010253
×	1.0000087	38
pr.	1.9999959	3010291
×	1.0000022	9
pr.	2.0000002	3010300

then $\frac{3010300}{10000000} = .3010300$ the log. 2.

Here the neareſt number to 2 in the Table, is 1.7782794, which I ſet down, and the index of x, 2500000. Then the neareſt multiplier to produce 2, is 1.1547819, which I ſet down and its index of x, 625000. And below I ſet the product 2.0535247; and the ſum of the indices 3125000. Then this laſt number being greater than 2, I find the diviſor 1.0366329, which I ſet down, and the index of x, 156250. And below I ſet the quotient 1.9809567; and the difference of the indices 2968750. The next multiplier I find to be 1.0090350, which I ſet down, and its index 39062. Then

Then I ſet down the product 1.9988546; and the ſum of the indices 3007812. And thus I proceed till I get 2 and the index 3010300; and thence the log. 3010300.

To get the multipliers as I proceed, I divide 2 by 1.77, and $\frac{2}{1.77} = 1.13$, and the neareſt number in the Table is 1.1547819. To get the diviſor, $\frac{2.053}{2} = 1.026$, and the neareſt number is 1.0366329. To find the next multiplier, $\frac{2}{1.9809} = 1.009$, and the diviſor is 1.0090350. And ſo on to the laſt.

Where note, a greater number (than 2.) requires a diviſor, and a leſs, a multiplier.

EXAM. 2.

To find the log. of a great number as 37.

Firſt find the log. of 3.7 as before, which is 0.5682017; and then the log. 37 is 1.5682017.

SCHOL. 1.

On the contrary, the number may be found by the ſame Table, anſwering to the given logarithm; that is done by continually approximating to the given index, as before you did to the number. Which is done, by continually adding (or ſubtracting) ſuch indices, as to make the ſum (or difference) the neareſt the given index; and at the ſame time multiplying (or dividing) the numbers.

EXAM.

What is the number of the log. 2.5406725.

First find it for .5406725, thus

Number.	Index.
3.1622776	5000000
1.0746078	312500
3.3982080	5312500
1.0181517	78125
3.4598912	5390625
1.0045073	19531
3.4754858	5410156
1.0005623	2441
3.4735327	5407715
1.0002811	1221
3.4725561	5406494
1.0000351	152
3.4726785	5406646
1.0000175	76
3.4727392	5406722
1.0000005	2
3.4727410	5406724

So that 3.4727410 is the number of the log. .5406724; and therefore 347.27410 is the number of the log. 2.5406725. But in such cases the general Table ought to be made to more places of figures.

In this example, I take the nearest index 5000000, and its number 3.1622776. Then I take the index 312500, as the nearest to 406724; and I set it down and its number 1.0746078. Then adding the indices, we have 5312500; and multiplying the numbers, gives 3.3982080. Then I take the index 78125 (the nearest to .0094225, what

what is wanting to make up 5406725), and its number 1.0181517. Then I get the index 5390625 and its number 3.4598912. And ſo proceed to to end.

SCHOL. 2.

Inſtead of making a Table, by taking the ſquare roots of 10 continually, one may uſe other roots, as the cube root, &c. But in working by it, the numbers will not converge ſo faſt. And obſerve, the nearer 1, the index of the root is, the faſter the numbers will converge, but there will be more labour in making the Table. Thus inſtead of $\frac{1}{2}$ (the index of the ſquare root) take $\frac{2}{3}$ or rather $\frac{3}{4}$ for the index, for making the Table. In working with $\frac{3}{4}$, each number muſt be cubed, and the ſquare root twice extracted, to get the next ſucceeding number in the Table. In ſuch a Table the differences are leſs, which makes the numbers approximate faſter. The index $\frac{7}{8}$ or rather $\frac{9}{10}$ would be exceeding good, but for the labour of calculating the Table. In $\frac{9}{10}$ the cube of every number muſt be cubed, and the ſquare root of the 5th root taken, which would be an immenſe labour. But then ſuch a Table will ſerve for all numbers; and it will converge 6 or 7 times as faſt as the ſquare roots.

PROB. II.

To find the relation between the indefinite root of any number, and its logarithm.

The indefinite root of any number is 1 + ſome very ſmall quantity, as $1 + x$; and if m be that root, then $1 + x^m$ may repreſent any number, m being indefinitely great. Let N be that number,

then $\overline{1+x}^{m} = N$, or $N^{\frac{1}{m}} = 1 + x$, whence $x = N^{\frac{1}{m}} - 1$. But by the property of the logarithmic curve, if t be the ſubtangent, then $mt \times N^{\frac{1}{m}} - 1 = \log. N$; where if $t = 1$, it will be Neper's Logarithm; if $t = .43429448$, it is the common logarithm.

Cor. 1. *If* r *be any other indefinite root,* $rt \times N^{\frac{1}{r}} - 1 = log.$ N.

Cor. 2. $m \times N^{\frac{1}{m}} - 1 = r \times N^{\frac{1}{r}} - 1$; *for they are both equal to* $\frac{log. N}{t}$.

Cor. 3. *If* P *be any other number, then* $N^{\frac{1}{m}} - 1 : P^{\frac{1}{m}} - 1 :: log.$ N : *log.* P.

Cor. 4. $m \times N^{\frac{1}{m}} - 1 : r \times P^{\frac{1}{r}} - 1 :: log.$ N : *log.* P.

Cor. 5. *Log.* $N = \frac{N^{\frac{1}{m}} - 1}{10^{\frac{1}{m}} - 1}$. *This follows from Cor.* 3. *by putting* $P = 10$, *and its log.* $= 1$.

PROB. III.

To make logarithms by the indefinite roots of the numbers.

From the laſt Prob. or from Cor. 3. the logarithm of a number N may be found, if its indefinite

finite root can but be extracted, which indeed is all the difficulty.

To find the log. N.

Involve 2 to ſome power large enough for the index, as ſuppoſe $2^{23} = 8388608 = m$, which is to be made uſe of for all numbers N. Then $mt = 8388608 \times .43429448 = 3643126$. Then extract the ſquare root of N 23 times, which will be $N^{\frac{1}{m}}$, then $mt \times N^{\frac{1}{m}} - 1 = \log.$ N, for the common log.

EXAM.

To find the log. 2.

Here $N = 2$, and $N^{\frac{1}{m}} = 1.00000008262958 79$; and $N^{\frac{1}{m}} - 1 = .0^{7}8262958 79$ (0^{7} denoting 7 cyphers), then $mt \times N^{\frac{1}{m}} - 1 = 3643126. \times .0^{7}8262958 79 = .3010300$ for the log. 2.

Or thus.

Let $\overline{1 + x}^{8388608} = N = 2$. Then by the method of extracting the roots of adfected equations, taking the 3 firſt terms only, we ſhall have $1 + 8388608x + \frac{8388608 \times 8388607}{2} x^2$ &c. $= 2$, or $x + 4194303\frac{1}{2}xx = \frac{1}{8388608}$, whence $x = \frac{.0^{13}284}{.0^{7}23 + e} = .078$; and repeating the operation once or twice, at laſt $x = .0^{7}8262958 79$; $= N^{\frac{1}{m}} - 1$, and $mtx = .3010300$, as before, for the log. 2. But involving the quantities here is extremely difficult;

cult; and beſides, it does not anſwer in large numbers, but diverges.

Or thus.

Extract the ſquare root of 10, for 23 extractions, then $10^{\frac{1}{m}} = 1.0000002744895^{7}$, and $10^{\frac{1}{m}} - 1 = .0^{6}27448954$; then by Cor. 5. Prob. II. log. $N = \frac{N^{\frac{1}{m}} - 1}{.0^{6}27448954}$.

EXAM.

To find the log. 2.

Here $N = 2$, and $N^{\frac{1}{m}} - 1 = .0782629587^{9}$, therefore $\frac{.0782629587^{9}}{.0^{6}27448954} = .3010300$ for the log. 2.

If you chuſe to have 10000000 &c. for the index *m*, you muſt extract the 5th root, and then the ſquare root of that, which gives the 10th root. Again, extract the 5th root of this, and then the ſquare root, for the 100th root; and proceed thus as often as you have cyphers in the index, which here requires 7 repetitions; and the laſt root will be $1 + x$. Or elſe extract the 10th root at once, by Rule III. page 269, *Algebra*; which is the quickeſt way.

It is evident all theſe methods are impracticable, from the great labour required in extractions and involutions. Yet as difficult as it is, the method, by continually extracting the ſquare root, was that made uſe of, at the firſt invention of logarithms. But, however, theſe methods ſhew us very clearly the nature and conſtitution of logarithms, and the dependence they have on the numbers themſelves.

ART.

ART. VIII.

The Interpolation of Quantities.

PROB. I.

To inveſtigate a general rule for the interpolation of quantities.

LET m, n, p, q, &c. be a ſeries of numbers, and $a\ b\ c\ d$, &c. a ſeries of quantities, among which another is to be interpolated. Let x be the number in the firſt ſeries, and y the number to be interpolated in the ſecond ſeries.

Aſſume the powers of x, affected with unknown coefficients, to repreſent y; thus, let $y = g + bx + kxx + lx^3$ &c. then to determine g, b, k, l, &c. we muſt take

$$g + bm + km^2 + lm^3 \text{ &c.} = a.$$
$$g + bn + kn^2 + ln^3 \text{ &c.} = b.$$
$$g + bp + kp^2 + lp^3 \text{ &c.} = c.$$
$$g + bq + kq^2 + lq^3 \text{ &c.} = d, \text{ &c.}$$

Here having as many equations, as unknown coefficients, g, b, k, l; theſe coefficients will be determined. Hence for 4 quantities we get,

$$l = \frac{a \times \overline{q-p} \cdot \overline{p-n} \cdot \overline{q-n} - b \times \overline{q-p} \cdot \overline{q-m} \cdot \overline{p-m} + c \times \overline{n-m} \cdot \overline{q-m} \cdot \overline{q-n} - d \times \overline{p-m} \cdot \overline{p-n} \cdot \overline{q-m}}{\overline{p-m} \cdot \overline{n-m} \cdot \overline{q-p} \cdot \overline{p-n} \cdot \overline{q-m} \cdot \overline{n-q}}.$$

$$k = \frac{a \times \overline{p-n} - b \times \overline{p-m} + c \times \overline{n-m}}{\overline{p-n} \cdot \overline{p-m} \cdot \overline{n-m}} - l \times \overline{m+n+p}.$$

$$b = \frac{b-a}{n-m} - l \times \overline{mm+nn+mn} - k \times \overline{m+n}.$$

$$g = \frac{an-bm}{n-m} + lmn \times \overline{n+m} + kmn.$$

Therefore all the numbers and quantities m, n, p, q, and a, b, c, d, being known, g, b, k, l, will be known.

Then aſſume any value for x, according to the place it is to have in the ſeries m, n, p, q; and the correſpondent value of y will be known, in the like place, among a, b, c, d: that is, $y = g + bx + kxx + lx^3$.

Cor. *If only* 3 *quantities are concerned, as* m, n, p, *and* a, b, c. *And* $y = g + bx + kxx$; *then*

$$k = \frac{\overline{a.p-n} - \overline{b.p-m} + \overline{c.n-m}}{\overline{p-n.p-m.n-m}}$$

$$b = \frac{b-a}{n-m} - k \times \overline{m+n}$$

$$g = \frac{an-bm}{n-m} + kmn.$$

SCHOL.

Some other methods of interpolation may be ſeen in my Treatiſe of the Differential Method.

PROB. II.

To find a rule for the interpolation of a quantity, among 4 *quantities, at equal diſtances.*

Suppoſe the numbers (in the laſt Prob.) m, n, p, q, to be 0, 1, 2, 3. Then by ſubſtituting numbers for theſe letters, we ſhall have

$$l = \frac{2a-6b+6c-2d}{-12} = \frac{\overline{d-a+3 \times \overline{b-c}}}{6}$$

$$k = \frac{a-2b+c}{2} - 3l$$

$$b = b - a - l - k$$

$$g = a.$$

All

All theſe methods are the ſame with M. De la Cail's, but are very laborious,

Otherwiſe thus.

Suppoſe as before a, b, c, d, are equidiſtant, and that x lies between b and c. By Prop. V. *Differential Method*, $y = A + xB + x.\frac{x-1}{2}C + x.\frac{x-1}{2}.\frac{x-2}{3}D$, or putting $1+v$ for x, and a, $\overline{d}$, $\overline{d'}$, $\overline{d''}$, for A, B, C, D; then $y = a + \overline{1+v}.\overline{d} + \frac{\overline{1+v.v}}{2}\overline{d'} + \frac{\overline{1+v.v.v-1}}{6}\overline{d''}$. But $b + \overline{c-b}.v$ is the value of y by the mean motion, and $b = a + \overline{d}$, and $c - b = \overline{d} + \overline{d'}$, then this value of y is $a + \overline{1+v}.\overline{d} + v\overline{d'}$; this ſubtracted from the true value of y, leaves $\frac{\overline{1+v.v}}{2}\overline{d'} - v\overline{d'} + \frac{\overline{1+v.v.v-1}}{6}\overline{d''} = \frac{\overline{v.v-1}}{2}\overline{d'} + \frac{\overline{v.v-1}.\overline{1+v}}{6}\overline{d''}$ for the correction $= \overline{d'} + \frac{\overline{1+v}}{3}\overline{d''} \times \frac{\overline{v.v-1}}{2} = \frac{\overline{v.v-1}}{2} \times \overline{-d'' - \frac{1+v}{3}d''}$.

But (Cor. 1. Prop. II. Diff. Method).

$b = a + \overline{d}$

$c = a + 2\overline{d} + \overline{d'}$

$d = a + 3\overline{d} + 3\overline{d'} + \overline{d''}$, whence

$b + c = 2a + 3\overline{d} + \overline{d'}$

$a + d = 2a + 3\overline{d} + 3\overline{d'} + \overline{d''}$, ſubtracted

$\overline{b+c} - \overline{a+d} = -2\overline{d'} - \overline{d''}$,

and $\frac{\overline{v.1-v}}{2} \times \frac{\overline{b+c} - \overline{a+d}}{2} = \frac{\overline{v.1-v}}{2} \times \overline{-d' - \frac{d''}{2}}$,

and this is extremely near the correction required, and

and when $v = \frac{1}{2}$, it is exactly so. Therefore if M = value of y by mean motion, $M + \frac{v \times \overline{1-v}}{2} \times \frac{\overline{b+c} - \overline{a+d}}{2} = y$ required, where $M = b + \overline{c-b}.v$.

Cor. 1. *If there be only three equidistant quantities, and d wanting; then* $y = M + \frac{v.\overline{1-v}}{2} \times \frac{b+c-a}{2}$.

Cor. 2. *If there be only three equidistant quantities, and a wanting; then* $y = M + \frac{v.\overline{1-v}}{2} \times \frac{b+c-d}{2}$.

EXAM.

Four longitudes of the moon being given, as follows; to find it at $16^h\ 22'\ 16''$. July 16, 1767.

July 16. Noon	11	29°	29′	34″	$= a$
Midnight	0	6	40	25	$= b$
17 Noon	0	13	47	48	$= c$
Midnight	0	20	51	27	$= d$

$7° 7' 23''$

Here $a = -30' 26''$, then $c - b = 7° 7' 23''$, and $v = 4^h\ 22' 16''$, and $1 - v = 7^h\ 37' 44''$; and $M = 0^s\ 6°\ 40'\ 25'' + \frac{7° 7' 23'' \times 4^h\ 22' 16''}{12} = 0^s\ 9°\ 16'\ 6''$.

$b + c = 20°\ 28'\ 13''$.

sub. $a + d = 20\ 21\ 1$

$+ \quad 07\ 12$

$v.\overline{1-v} = \frac{4\ 22\ 16}{12} \times \frac{7\ 37\ 44}{12} = .2316$

and $\frac{.2316}{2} \times \frac{7'\ 12''}{2} = 25''$

Therefore $y = M + 25'' = 0^s\ 9°\ 16'\ 31''$,

but

but the beſt way is to turn the quantities into decimals.

PROB. III.

Five or more quantities a, b, c, d, e, being given at equal diſtances; to interpolate a quantity between b and c; or between c and d.

By Prop. V. of the Differential Method, we have $y = A + xB + x.\frac{x-1}{2}C + x.\frac{x-1}{2}.\frac{x-2}{3}D + x.\frac{x-1}{2}.\frac{x-2}{3}.\frac{x-3}{4}E$. Put $1+v=x$, and d', d'', d''', $d''''=B$, C, D, E. Then $y = a + \overline{1+v}.d' + \frac{\overline{1+v}.v}{2}d'' - \frac{\overline{1+v}.v.\overline{1-v}}{6}d''' + \frac{\overline{1+v}.v.\overline{1-v}.\overline{2-v}}{24}d''''$, between b and c.

But if y is to ſtand between c and d, put $2+v=x$, and then $y = a + \overline{2+v}.d' + \frac{\overline{2+v}.\overline{1+v}}{2}d'' + \frac{\overline{2+v}.\overline{1+v}.v}{6}d''' - \frac{\overline{2+v}.\overline{1+v}.v.\overline{1-v}}{24}d''''$. where d', d'', d''', d'''', are the firſt, of the firſt, ſecond, third, and fourth orders of differences. In 6 quantities, put alſo $x=2+v$.

They that want to ſee more of this matter, may conſult the ſaid Differential Method, where he will find general ſolutions to all queſtions of this nature, whether the quantities ſtand at equal or unequal diſtances.

ART.

ART. IX.

Of finding the Longitude at Sea.

HAVING in my book of Aſtronomy given a ſhort ſketch of a method for finding the longitude, by the moon's diſtance from a ſtar; I ſhall here proſecute the ſubject further, and compleat that method; and alſo lay down ſome other rules of like ſort, by which the ſame thing may be performed. But I do not adviſe ſuch methods to be commonly practiſed at ſea, for they take too much labour and calculation; and beſides we are not to expect that any method by the moon, can be ſo exact as one could wiſh, owing to her ſlow motion from the ſun; being only 12 degrees in a day out of 360. If we were bleſſed with ſuch a ſatellite as Jupiter, that revolves in leſs than two days, nothing would be more eaſy. But as the caſe ſtands, we cannot reaſonably expect ſuch accuracy, without the moſt laborious calculations, and an incredible degree of care and exactneſs in making obſervations; and even the frequent repeating of them. At this rate ſome people ought to have nothing elſe to do a ſhip-board; but there is ſeldom occaſion there for ſuch ſupernumerary hands. The right uſe then of theſe methods is in time of diſtreſs, when a ſhip, by ſtreſs of weather, has loſt her way, and cannot purſue her reckoning. In ſuch a caſe as this, ſuch a method becomes uſeful; for if the ſhip's longitude can be found this way to a degree or two, it will generally anſwer the ſame purpoſe, as if they had it to a minute or two.

It would not be difficult to make Tables to facilitate the calculations, as well in this as in other methods,

methods, but that would not be worth the while, as the method is to be uſed ſo ſeldom. And when that happens, a man need not grudge the labour of 3 or 4 hours, to ſet the ſhip right; which may not happen above once in a voyage. To do it often would be ſlavery, indeed, when the common known methods are ſufficient to guide a ſhip in moderate weather. And therefore the ſeamen will never thank us for impoſing ſuch a taſk upon them; nay, they will curſe us for it, as being a thing as needleſs and uſeleſs, as taking in the ſpheroidical figure of the earth. And yet in the time of danger and diſtreſs, when the reckoning is loſt, and when a ſhip is rambling ſhe knows not where; they will be glad of ſuch a rule, to ſet them right; and this is all the uſe it can have. Fig.

PROB. I.

Having the apparent diſtance of the moon from a ſtar, and their zenith diſtances; to find the true diſtance.

Let M be the apparent place of the moon, and S that of the ſtar; and *m*, *s*, their true places; where M*m* is the moon's parallax leſſened by the refraction, and S*s* is the ſtar's refraction, both which are known, for the known altitudes. And let Z be the zenith. 31.

Then in the ſpherical triangle MZS; there is given the diſtance MS, and the ſides ZM, ZS, the co-altitudes; to find the angle Z, by Caſe 11th of Spherical Trigonometry.

The angle Z being found; in the ſpherical triangle *m*Z*s*, there are given the ſides Z*m*, Z*s*, and the angle Z; to find the diſtance *ms*, by Caſe 8th of Spherical Trigonometry; and *ms* will be the true diſtance.

SCHOL.

Fig.

SCHOL.

31. In Prob. XVII. Sect. VI. Aſtronomy, I have given a ſolution to this, by approximation. And in the Appendix, an accurate method by the natural ſines; and I believe a ſhorter method cannot be found.

Note, the difference of the ſides MS, *ms*, can never amount to a degree.

PROB. II.

To find the longitude of a ſhip at ſea.

1. To perform this, we muſt have Tables ſhewing the right aſcenſion and declination of the ſun and moon, to minutes and tenths, every day at noon, made for a certain place as *Greenwich*; and likewiſe of the principal fixt ſtars, that lie near the moon's way.

2. At the place propoſed, the moon's diſtance from ſome known ſtar, muſt be truly taken with a good inſtrument; and alſo their altitudes at that time, and the exact time noted down, which (reduced to degrees) is the ſun's weſting from the ſhip.

3. The altitudes muſt be cleared of refraction and parallax, and the apparent diſtance reduced to the true diſtance, by Prob. I.

4. Take the longitude as near as you can by the dead reckoning, and convert it into time, which reckon weſt from *Greenwich*, becauſe the ſun's apparent motion is weſtward. Add this time to the time of obſervation, gives the hour (*h*) at *Greenwich*, or the time elapſed ſince noon at *Greenwich*, (rejecting 24 hours, if it exceeds). Reduce it to the decimal of a day, which put $= t$, and the remainder of the day $= v$, and get tv.

5. Lay the Tables before you, and take the moon's right aſcenſion for four days; two before, and

and two after, the time at *Greenwich*, which call A, B, C, D. Then ſubtract the ſum of the extremes (A, D), from the ſum of the means (B, C), and call the remainder F. Then the mean change of right aſcenſion, in the time t, is $= t \times \overline{C - B}$; and the true change $= t \times \overline{C - B} + \frac{tv}{4} F = y$. And the moon's true right aſcenſion $= B + t \times \overline{C - B} + \frac{tvF}{4} = B + y$. Fig. 31.

6. In like manner, for the moon's declination take four days declination; two before, and two after the time, which call a, b, c, d; and theſe of a contrary denomination muſt be marked negative. Then ſubtract the ſum of the extremes (a, d), from the ſum of the means (b, c), which call f. Then the declination by mean motion, is $= b + t \times \overline{c - b}$, and the true declination $= b + t \times \overline{c - b} + \frac{tv}{4} f$.

7. Then having the diſtance of the moon and ſtar, and their polar diſtances, find the angle at the pole, (by Caſe 11th of Spherical Trigonometry), which is the difference of right aſcenſions of the moon and ſtar. This, with the ſtar's right aſcenſion, gives the moon's preſent right aſcenſion.

8. The moon's preſent right aſcenſion, and her right aſcenſion at noon (by the Tables), being now known; we ſhall have the right aſcenſion gained (G), in the time t, or b hours.

9. Then $\frac{Gb}{y}$ = ſun's weſting from *Greenwich* in hours; from this ſubtract the time at the ſhip; the difference, turned into degrees, is the longitude of the ſhip.

10. If

Fig. 10. If more exactneſs be required, either take the laſt reſult for a new operation, or elſe the nearest round number, for the eaſe of calculation; and find t and b anew; and then repeat the operation. But if the longitude was near the truth, $\frac{1}{4}tvF$ and $\frac{1}{4}tvf$, may remain the ſame, as alſo g and b. And when moſt of the quantities remain the ſame, a new operation will ſoon be done.

30. 11. The moſt expeditious way of working, is to uſe decimals for all the quantities. And to give a true notion of the work, always draw a figure. Here S⊙ is the time at the ſhip, G⊙ the time at *Greenwich*, and GS the ſhip's longitude.

This method may be equally applied to the ſun, as well as to a ſtar; but with ſomething more trouble. For the change of his right aſcenſion and declination muſt alſo be found.

Note, the proceſs, Art. 5 and 6, is founded on Prob. II. of the laſt article.

II. *Otherwiſe thus.*

Here we muſt have Tables of longitudes and latitudes of the moon and ſtars. And inſtead of the right aſcenſion and declination of the ſun, moon, and ſtars; make uſe of their longitudes and latitudes; and work by the very ſame rule as before. Thus Art. 5. gives the moon's true longitude, and Art. 6. her true latitude. And Art. 7. finds the diff. longitude of the moon and ſtar, and conſequently the moon's preſent longitude. By Art. 8. is had the moon's longitude gained in the time t. And ſo on.

It is worth obſerving, that the very ſame rules will anſwer for the Tables of longitude and latitude, as well as for thoſe of the right aſcenſion and declination, *mutatis mutandis*. And it is the peculiar advantage of this method, that the problem may be ſolved either way; and therefore it may be done by

by ſome of the common almanacks, provided their numbers be true and correct. Fig. 30.

SCHOL. 1.

When the moon has little or no declination, one need not find the quantity F nor *f*, nor the correction.

In addition and ſubtraction, the ſigns of the quantities muſt be carefully obſerved. And therefore when a greater quantity is to be ſubtracted from a leſſer; the remainder muſt be ſet down negative, as may happen in Art. 5 and 6. Alſo in Art. 9. when the time at the ſhip is greater; the remainder is negative, and the longitude eaſt.

When the ſhip lies eaſt of *Greenwich*, it may do as well to reckon that diſtance negative.

Note, moſt of the work will be beſt done by logarithms.

III. *Otherwiſe thus.*

1. From the aſſumed longitude and given time of obſervation, having got the moon's right aſcenſion, by Art. 5. from the Tables;

And likewiſe having got her right aſcenſion by calculation, by Art. 7. compare them together. If they are the ſame, you may be ſure your longitude is right: if they differ, note the error.

2. Make a nearer ſuppoſition for the longitude, as you may eaſily do. Then having found the moon's right aſcenſion twice (by Art. 5. and 7.) as before; if they differ, note the error again.

Then by the method of trial and error, the true longitude of the ſhip will be found.

And the ſame thing may be done, by making uſe of the moon's longitude (inſtead of her right aſcenſion), according to Rule II.

Note, one may eaſily know, as ſoon as the firſt operation is over, by comparing the right aſcenſions, &c. whether the firſt longitude be taken too

Fig. 30. great or too little; and take the next accordingly. This Rule comes very near.

SCHOL. 2.

After the mean proportional part of the declination is found; if you would not take the trouble of finding F or f, take $\frac{1}{3}$ the *degrees* of declination; and add (always) so many minutes (z) to the declination; and $\frac{1}{3}$ of that to or from the (mean) right aſcenſion. Or more exactly, for 0, 3, 6, or 9 hours before or after noon, take 0, $\frac{4}{10}$, $\frac{7}{10}$, or $\frac{9}{10}$ of the greateſt correction (z), to be added (always) to the mean declination. When the declination decreaſes, add; when it increaſes, ſubtract, the correction to or from the right aſcenſion.

IV. *Or thus.*

1. From the ſuppoſed longitude, and given time of obſervation, proceed by the firſt Rule, to get the moon's right aſcenſion (m) by the Tables; and likewiſe her right aſcenſion (n) by calculation; and find the angle at the moon. Then,

2. Put τ = cotan $<$ at the moon.
s = ſine moon's polar diſtance.
$q = c - b$; $p = C - B$, then
$\frac{6 \times \overline{n - m}}{p - \frac{\tau}{s} q}$ = correction of the longitude of the ſhip in degrees; to be added or ſubtracted, according as the ſign directs.

Note, p and q are in degrees, but $n - m$ is in minutes; and τ is negative, when the angle is greater than 90 degrees.

For let $t + x$ be the true time, x being the increment of t, then qx = variation of declination, or of the polar diſtance. And (by Cotes, Theor.

23,

23, pa. 17.) $s : \tau :: qx : \frac{\tau}{s} qx =$ variation of right ascenſion. Then $m + px = n + \frac{\tau}{s} qx$, and $px - \frac{\tau}{s} qx = n - m$ in degrees; or $px - \frac{\tau}{s} qx = \frac{1}{60} \times \overline{n - m}$ ($n - m$ being minutes); therefore $x = \frac{\frac{1}{60} \times \overline{n - m}}{p - \frac{\tau}{s} q}$, the increment of time in days, and $\frac{24}{60} \times \frac{n - m}{p - \frac{\tau}{s} q} =$ incr. time in hours, and $\frac{24 \times 15}{60} \times \frac{n - m}{p - \frac{\tau}{s} q}$ or $\frac{6 \times \overline{n - m}}{p - \frac{\tau}{s} q} =$ increment of the degrees of longitude. Fig. 30.

This is eaſily wrought by logarithms; and this correction will come very near, when the ſuppoſed longitude is within a few degrees of the truth.

This is as eaſily done by making uſe of the moon's longitude, inſtead of her right aſcenſion. But then the angle at the moon muſt be found in the triangle whoſe two ſides are the co-latitudes of the moon and ſtar, inſtead of the co-declinations.

EXAM. I.

On April 4th, 1767, at 4h 28m $\frac{1}{3}$ apparent time, the diſtance of the moon from the ſun was obſerved to be 74° 16′, and altitude of the ſun 22° 18′, of the moon 86° 53′. The longitude by the reckoning 17 46 W (= 1h 11m in time.) To find the ſhip's true longitude.

Here $Ss = 3'$, $Mm = 9'$. Therefore there is given, ZM, ZS, MS, and Zm, Zs; then by Prob. I. ms is found = 74° 12′. 31.

Fig.

Preparation.

By Rule I.

30. Long time $1^h\ 11^m$
time of obſervat. $4\ 28\frac{1}{3}$

$5\ 39\frac{1}{3} = 5.655 = b$

$\frac{5.655}{24} = .2356 = t$
$.7644 = v$ $\Big\}\ tv = .18,$

$A =$	$71°.050$	$a =$	$26°.483$ N
$B =$	85.533	$b =$	26.783
$C =$	99.483	$c =$	25.650
$D =$	112.706	$d =$	23.300
$B+C =$	185.016	$b+c =$	52.433
$A+D =$	183.766	$a+d =$	49.783
$F =$	1.250	$f =$	2.650
$C-B =$	13.950	$c-b =$	-1.133
$C-B \times t =$	3.286	$t \times c-b =$	$-.266$
$\frac{1}{4}tvF =$	$.056$	$\frac{1}{4}tvf =$	$.119$
$y =$	3.342	$b =$	26.783
$B+y =$	88.875	☽'s true declin.	26.636
the ☽'s right aſcenſion.		☽'s pol. diſt. =	63.304
			$63°\ 22'$

⊙'s change of declin. in a day is	$22'.8$
and $.235 \times 22.8 =$	5.3
⊙ decl. Ap. 4th	$5\ 42.8$
⊙ preſent decl.	$5\ 48.1$

Change ⊙ *r.* aſc. in a day,	$54'\frac{1}{2}$
and $.2354 \times 54.5 =$	12.8
⊙ *r.* aſcenſ. Ap. 4.	$13\ 19.5$
⊙ preſent *r.* aſc.	$13\ 32.3$

Operation.

Fig. 32.

Operation.

Baſe, ⊙ diſt. from ☽ 74° 12

S. ☽ polar diſt.	63 22,	co. ar.	0.0487142
S. ⊙ polar diſt.	84 12,	co. ar.	0.0022290
diff. ſides	20 50		
+ baſe	95 02		
half	47 31, S.		9.8677466
— diff. ſides	26 41, S.		9.6523035
			2)19.5709933
	S. 37 36.3		9.7854966
	2		
∠P =	75 12.6	diff. right aſcen.	
	13 32.3	⊙ *r.* aſcenſion	
	88 45	☽ *s r.* aſcen. now	
	85 32	☽ *r.* aſ. at noon	
G =	3 13	*r.* aſ. gained	
or	3.216		

then $\frac{3.216 \times 5.655}{3.342} = 5^h.44 = 5^h\ 26^m$

And $5^h\ 26^m$ ⊙ weſt from Greenwich
— 4 28 time at the ſhip

0 58 = $14\frac{1}{2}$ deg. the longitude weſt.

The ſame Ex. by Rule II.

The quantities *ms*, *h*, *t*, *v* and *tv* remain the ſame; then,

Fig. 30.

Preparation.

A =	73.053	a =	4.130 N
B =	86.006	b =	3.364
C =	98.556	c =	2.454
D =	110.782	d =	1.454
B+C =	184.562	$b+c$ =	5.818
A+D =	183.835	$a+d$ =	5.584
F =	.727	f =	.234
C—B =	12.550	$c-b$ = —	.910
C—B × t =	2.957	$t \times c-b$ = —	.215
$\frac{1}{4}tvf$ =	0.033	$\frac{1}{4}tvf$ =	.105
y =	2.990	b =	3.364
B + y =	88.996	☽'s true lat.	3.256
☽'s longitude		☽'s co. lat.	86.744
		or	86 44½

Operation.

32. In the quadrantal triangle MPS, where P is the pole of the ecliptic, and PS the sun's polar distance = 90°; MP the ☽'s colat. 86° 44′½; MS the distance of the sun and moon 74° 12′. To find the ∠ P.

Cos. Mn or S. 86 44½	9.9991473
rad.	10.
cos. 74 12	9.4350161
cos. nS, or cos. P, 74° 10′	9.4358688
then dif. lon. of ☉ and ☽ =	74° 10′
☉'s long.	14 42½
☽ longit. now	88 52½
☽ long. at noon	86 00⅓
longit. gained	2 52 = 2.866

then $\frac{Gb}{y}$ = 5ʰ.42 = 5ʰ 25ᵐ ☉ west from Greenwich.

— 4 28 time at the ship.

0 57 = 14¼ the long. west.

If

If this had been wrought by mean motion, we should have, the moon's right aſcenſion 88° 49′ Fig. 33.
the moon's declination 26 31
and the longitude 16 degrees.

EXAM. 2.

Sept. 5th, 1767, *at* 15h 7m⅓ *apparent time, the corrected diſtance of the moon from the ſtar* α *pegaſi was* 44° 0′⅓; *the longitude by the reckoning* 64° 32′ E (= 4h 18m *time*). *To find the true longitude.*

By RULE I.

Preparation.

4h 18m longitude time
— 15 7⅓ time at the ſhip.
— 10 49⅓ = *b*, hour at Greenwich.
or 10,822 = *b*,
and 10.822 = .4509 = *t* } *tv* = .2476
24 .5491 = *v* }

A =	290° 16′	*a* =	23° 13′ S
B =	305 5	*b* =	19 19
C =	319 31	*c* =	14 7
D =	333 35	*d* =	7 58
B+C =	624 36	*b*+*c* =	33 26
A+D =	623 51	*a*+*d* =	31 11
	0 45	*f* = +	2 15
or F =	.75	or *f* =	2.25
C—B =	14 26	*c*—*b* = —	5 12 = —5.2
or =	14.433	*c*—*b*×*t* = —	2.345
C—B × *t* =	6.508	½*tvf* =	.138
½*tv*F =	.046	*b* =	19.316
y =	6.554	☽'s true decl.	17.109
B =	305.083		17 6.7
B+*y* =	311.637	☽'s polar diſtance }	107° 6.7′

the ☽'s right aſcenſ.

Fig. 34.

Operation.

Baſe, or ✷ diſt. from ☽,	44	0½		
S. ☽'s polar diſt.	107	6.7	coar.	0.0196621
S. ✷'s polar diſt.	76	2	coar.	0.0130330
diff. ſides	31	4.7		
+ baſe	75	5		
half	37	32.5, S.		9.7848584
— dif. ſides	6	27.8, S.		9.0514496
				2)18.8690031
S.	15	46.8		9.4345015
		2		
∠P =	31	33.6	diff. right aſcenſ.	
	343	18	✷ right aſcenſ.	
	311	44.4	☽'s right aſ. now	
	305	5	☽'s *r.* aſc. at noon	
G =	6	39.4	☽'s *r.* aſ. gained.	
=	6.656			

then $\frac{Gb}{y}$ = 10ʰ.99 = 10ʰ 59ᵐ.3, ⊙ weſt from Greenwich.

—	15	7.3	time at the ſhip.
	4	8	= 62° the longi. E.

The longitude by mean motion comes out 63° 45′.

Again, for another operation.

Aſſume 60 for the longitude of the ſhip = 4 hours.

Preparation.

Fig. 34.

Preparation.

$4^h\ 0$ ſuppoſed time
$-\ 15\ 7\frac{1}{2}$ time at ſhip

$11\ 7\frac{1}{2} = 11.122 = b$, the time at Greenwich,

and $\frac{11.122}{24} = .4634 = t$, $.5366 = v$; $tv = .2486$

$t \times \overline{C-B} =$	6.688	$t \times \overline{c-b} =$	$-$ 2.409
$B =$	305.083	$\frac{1}{4}tvf =$	.138
$\frac{1}{4}tvF =$	.046	$b =$	19.317
$B+y =$	311.817		17.046
☽'s right aſcenſ.		the ☽'s declination.	
and $y =$	6.734,	☽'s polar diſt.	$= 72°\ 57'$

Operation.

Baſe, or ☽ diſt. from ✷ 44 0.3

S. ☽'s polar diſt.	107	3,	coar.	0.0195197
S. ✷ polar diſt.	76	2,	coar.	0.0130330
diff. ſides	31	1		
+ baſe	75	1.3		
half	37	30.6,	S.	9.7845569
− dif. ſides	6	29.6,	S.	9.0534890
				2)18.8705986
S.	15	$48\frac{1}{2}$		9.4352993
		2		

$\measuredangle P =$ 31 37 dif. right aſcenſ.
343 18 ✷ right aſcen.

311 41 ☽ right aſcen. now
305 5 ☽ right aſ. at noon
$G =$ 6 36 ☽ right aſ. gained
or 6.6

then

Fig. 84. then $\frac{Gb}{y}$ = 10^h.9 ⊙ weſt from Greenwich.

— 15^h 7.3^m = — 15.12 time at the ſhip.

4.22 the long. in hours E.

4.22 × 15 = 63°.3 = 63° 18′ the longitude E. in degrees.

Calculation of the ſame for 62° 40′ (= 4^h 10^m.6)

Preparation.

^{h}	^{m}	
4	10.6	ſuppoſed time
— 15	7.3	time at the ſhip
10	56.7	= 10.945 = b, hour at Greenwich.

$\frac{10.945}{24}$ = .456 = t, tv = .248

$t \times \overline{C-B}$ = 6.581 $t \times \overline{c-b}$ = — 2.37

$\frac{1}{4}tvF$ = .047 $\frac{1}{4}tvf$ = .14

y = 6.628 β = 19.317

B = 305.08$\frac{1}{2}$ ☽'s declin. 17.087 = 17° 5′.2

B + y = 311.711′ ☽'s pol. diſt. 107 5.2

= 311 42.6

the ☽'s right aſcen.

Operation.

Baſe, ☽ diſt. from ✷ 44 0.3

S. ☽'s polar diſt.	107	5.2	coar.	0.0196038
S. ✷ polar diſt.	76	2	coar.	0.0130330
diff. ſides	31	3.2		
+ baſe	75	3.5		
half	37	31.7	S.	9.7847214
— diff. ſides	6	28.5	S.	9.0521923
				2)18.8695505
				9.4347752
S.	15	47.5		S.

S. 15 47.5
2

Fig. 34.

$\angle P =$ 31 25 diff. right aſcen.
343 18 * right aſcen.
311 43 ☽'s right aſ. now
305 5 ☽'s right aſ. at noon
$G =$ 6 38 right aſcen. gained
or $=$ 6.633

then $\frac{Gb}{x} = 10^{t}.95 =$ 10 57
$-$ 15 7.3
$-$ 4 10.3 $= 62° 35'$ the longitude E.

The ſame Exam. by Rule II.

Preparation.

Having found as before $t = .451$, $tv = .2476$, and $b = 10.882$, from the Tables of longitude and latitude, we ſhall have 35.

	°		°
$A =$	288.566	$a =$	$-$ 1.043 S
$B =$	302.847	$b =$	$+$ 0.242 N
$C =$	317.558	$c =$	1.548
$D =$	332.613	$d =$	2.772
$B+C =$	620.405	$b+c =$	1.790
$A+D =$	621.179	$a+d =$	1.729
$F =$	$-$ 0.774	$f =$	.061
$C-B =$	14.711	$c-b =$	1.306
		$c-b \times t =$	.588
$C-B \times t =$	6.634	$\frac{1}{4}tvf =$	.004
$+\frac{1}{4}tvF =$	$-$ .048	$b =$	.242
$y =$	6.586	☽'s true lat.	.834
$B+y =$	309.433	☽'s polar diſt.	89.166
the ☽'s longitude.		or	89° 10'

Operation:

Fig. 34.

Operation.

Baſe, or ✱ diſt. from ☽ — 44° 0.3

S. ☽'s co-lat 89 10 coar. 0.0000459
S. ✱'s co-lat. 70 35.3 coar. 0.0254155

diff. ſides 18 34.7
+ baſe 62 35
S. half 31 17.5 S. 9.7154976
— diff. ſides, S. 12 42.8 S. 9.3425671

2)19.0835261

S. 20 22.45 9.5417630
2

< P = 40 44.9 diff. lon. ☽ and ✱
350 14.5 long. ✱

309° 29'.6 ☽'s long. now
302 50.8 ☽'s long. at noon

6.38.8 longitude gained.
or 6.646 = G_2

then $\frac{Gb}{y}$ = 10ᵗ.92

— time at the ſhip — 15.12

4.20

and 4.20 × 15 = 63° the long. eaſt.

Otherwiſe thus, by Rule III.

1. Suppoſe (as in the firſt calculation) that the longitude is 64° 32', then

☽'s right aſcenſion is 311° 38'.2 by the Tables,
and her right aſcen. 311 44.4 by calculation.

difference + 6.2 long. too great.

2. Suppoſe

2. Suppoſe (as in the ſecond calculation) that the longitude is 60° 00′, then Fig. 34.

☽'s right aſcenſion is 311° 49′ by the Tables,
and her right aſcenſion 311 41 by calculation.

difference — 8 long. too little.

Then by the rule of falſe, $\frac{4°.533 \times 6.2}{6.2+8} = 1°.98$ = 1° 59′ the correction, and 64° 32′ — 1° 59′ = 62° 33′ for the longitude, where 4°.533 = 64 32 — 60.

EXAM. 3.

Jan. 13th, 1767, *at* 10h 27m¼, *the diſtance of the moon from regulus, when corrected, was* 46° 32′½. *To find the longitude.* 35.

As here is no longitude given, we ſhall aſſume 0 longitude, for the firſt trial.

Preparation.

long. in time 0h 0m
time of obſerv. 10 27¼
10 27¼ = 10.454 = *b*.

and $\frac{10.454}{24} = .4356 = t$, $.5644 = v$; $tv = .2458$

A =	83°	11′	*a* =	27	7 N
B =	96	59	*b* =	26	15
C =	110	13	*c* =	24	7
D =	122		*d* =	20	56
B+C =	207	12	*b*+*c* =	50	22
A+D =	205	59	*a*+*d* =	48	3
F =	1	13	*f* =	2	19
or F =	1.217		or *f* =	2.317	
C—B =	13	14	*c*—*b* =	— 2	8
or =	13.233		or =	— 2.133	

tX

Fig. 35. $\frac{1}{2}$ rad. C B = .875 | $\frac{1}{2}$ rad. b = .927
$\frac{1}{2}$ rad. F = .075 | $\frac{1}{2}$ rad. f = .142
y = 5.826 | b = 26.250
B = 96.983 | 25.465
102.809 | or 25 28
or 102 49 | ☽'s declination.
the ☽ right aſcenſion. | ☽ pol. diſt. 64 32

Operation.

36. Baſe, ☽ diſt. from ☉, 46° 82½′

S. ☽'s polar diſt.	64 32,	cosr.	0.044391o
S. ☉ polar diſt.	76 54,	cosr.	0.011451ꝸ
diff. ſides	12 22		
½ baſe	58 54½		
half	29 27, S.		9.691668ꝫ
— diff. ſides	17 5 S.		9.467996o
			2)19.215507ꝫ
S.	23 54½		9.607753ꝫ
	2		

< P = 47 49 dif. right aſcenſion
148 59 ☉ right aſcenſion

101 10 ☽ right aſcenſ. now
96 59 ☽ right aſ. at noon

G = 4 11 right aſcenſ. gained
or 4.183 = G

then $\frac{Gb}{y}$ = 7ᵐ.505 = 7ᵐ 30ˢ
— 10 27
— 2 57

and 2ᵐ 57ˢ = 44°¼ longitude E.

Then

Then for a second trial, take the long. 44°; or rather 40° E. here now t &c. being found, and the operation performed, the true longitude will come out $43\frac{1}{4}$ E. Pag. 36.

If only the mean motion had been used, the longitude would have come out 43° 0′.

Otherwise solved, by Rule IV.

In the foregoing Example, suppose the longitude 40° E. then

long. 40° E. $\pm$ — $2^h\ 40^m$
time at ship = 10 27.2

$7\ 47.2 = 7^h.787$

and $\frac{7.787}{24} = .3244 = t$, $.6756 = v$, $tv = .2192$.

Preparation.

having found 35.

$\overline{C-B} = 13.233 = p$ $\quad c-b = -\ 2.133 = q$

then $t \times \overline{C-B} =$	4.293	$t \times \overline{c-b} =$ —	.691
$\frac{1}{2}tvF =$	.066	$b =$	26.250
$y =$	4.359	$\frac{1}{4}tvf =$	.127
$B =$	96.983		25.686
	101.342	or	25 41
$m =$	101° 20′	☽'s declination.	
the ☽ right ascension.		☽'s polar dist.	64 19

Operation

Fig. 36.

Operation for the ☾ P.

Baſe SM, ☽ diſt. from ✱ 46 32½

☽ pol. diſtance 64 19 coar. 0.04518

✱ pol. diſt. 76 54 coar. 0.01146

diff. ſides 12 35

+ baſe 59 7½

half 29 34 S. 9.69323

— diff. ſides 16 59 S. 9.46552

2)19.21539

S. 23 54.3 9.60769

2

☾ P = 47 48½ diff. right aſcen.

148 59½ ✱ right aſcen.

n = 101 11 ☽ right aſcen.

Operation for the ☾ M.

S. SM (46 32½) coar. 0.13914

S. P (47 48½) 9.86976

S. SP (76 54) 9.98855

S. M (96 12) 9.99745

then $n = 101^\circ\ 11'$

$m = 101\ 20$

$n - m = -\ 9'$ $\quad 6 \times \overline{n-m} = -54$

$\tau = -\ .109$ $\quad p = 13.233$

$s = .901$ $\quad -\frac{\tau}{s} q = .255$

$q = -\ 2.133$ $\quad 12.978 = 13$ near

$p = 13.233$

$\frac{\tau}{s} q = .255$

$-\ 40\ 0$

$-\ 4\ 6$ $\quad \frac{54}{13} = -4.1 = -4^\circ\ 6'$

44 6 E. long.

But

But this method requires too much labour; therefore rather work by Rule III. Fig. 36.

SCHOL.

Any method of finding the longitude by the moon is clogged with many difficulties; as,

1. The uncertainty of the sun's declination, when an observation is made for getting the time.

2. The uncertainty of the ship's latitude, at the time of observation.

3. The difficulty of taking the distance of the moon exactly from the sun or a star, particularly the sun.

4. The inaccuracy of the Tables made use of in the calculations.

5. The danger of mistakes by the numerous calculations to be gone through by any of these methods.

ART. X.

Of Intereſt.

HERE I ſhall compute the intereſt, and amount, of any ſums or annuities for any time in general, as well as whole years, at both ſimple and compound intereſt; and alſo their preſent worths.

Let p = principal or preſent worth.
a = annuity or yearly rent.
s = amount or ſum of all the arrears.
T = whole time of continuance.
b = fractional part of a year.
t = the even years.
R = the ſum of 1*l.* and its intereſt for a year.
r = the intereſt of 1*l.* for a year.
hence $T = t + b$, and $R = 1 + r$.

PROP. I.

The principal, time, and rate of intereſt being given; to find the amount or arrear; at ſimple intereſt.

As $1 : r :: p : rp$, the intereſt of p for a year. And $1 : rp :: T : Trp$, the intereſt of p for the time T. Therefore $p + Trp =$ whole arrear at the end of the time T; whence $p+prT$ or $p+prt+prb =$ whole amount $= s$.

Cor. 1. $pTr =$ *intereſt of p for the time* T.

Cor. 2. $\frac{sTr}{Tr+1} =$ *the diſcount of s for the time* T, *whereof p is the rebate.*

Cor.

Cor. 3. *In equation of payments, time* $=$ $\frac{\text{sum of the discounts}}{\text{sum pres. worths} \times r}$.

PROP. II.

Annuity, time, and rate of interest being given; to find the amount or arrear, at the end of the time, at simple interest.

By Cor. 1. Prop. I.

bra = interest of the rent due at t years end.

$\overline{1+b}.ra$ = interest of the rent due at $t-1$ years end.

$\overline{2+b}.ra$ = interest of the rent due at $t-2$ years end.

$\overline{3+b}.ra$ = interest of the rent due at $t-3$ years end.

$\overline{4+b}.ra$ = interest of the rent due at $t-4$ years end.

and so on till

$\overline{t-1+b}.ra$ = interest of the rent due at 1 years end

Therefore if a be the rent, the sum will be $bra + \overline{1+b}.ra + \overline{2+b}.ra + \overline{3+b}.ra$ &c. to $\overline{n-1+b}.ra = b + \overline{1+b} + \overline{2+b} + \overline{3+b} \ldots \overline{t-1+b} : \times ra$, for the whole interest. But the series, consisting of $t-$ terms, its sum is $= \frac{t-1+2b}{2} \times tra$, to which add the sum of the rents Ta, then $s = \frac{t.\overline{t-1}+2bt}{2} \times ra + Ta$.

Cor. 1. $a = \frac{2s}{\overline{tt-t}+2bt \times r + 2T}$.

Cor. 2. $r = \frac{\frac{2s}{a} - 2T}{\overline{tt-t}+2bt}$.

Cor. 3. *To find* T, put $b = 0$, then $2s = \overline{tt-t}.ra + 2t$, and $tt + \frac{2-ra}{ra} t = \frac{2s}{ra}$, whence t will be

found, which will be ſomething too big, but the next leſſer whole number is to be taken. Then to find b, we have $2s-2ta-2ba=\overline{tt-t+2bt}\times ra$, and $2btra+2ba=2s-2ta+tra-ttra$, and $b=\frac{\frac{2s}{a}-2t+tr-ttr}{2tr+2}$; when this does not anſwer, put another whole number for t.

PROP. III.

Having given the annuity, time of continuance, and rate of ſimple intereſt; to find the preſent worth.

By Prop. I. $p+prT=s$, and by Prop. II. $\frac{\overline{tt-t+2bt}}{2}\times ra+Ta=s$; therefore $p+prT=\frac{\overline{tt-t+2bt}}{2}\times ra+Ta$, whence $p=\frac{\overline{\overline{tt-t+2bt}\times ra+2Ta}}{2rT+2}$.

Cor. 1. $a=\frac{\overline{2rT+2}\times p}{\overline{tt-t+2bt}\times r+2T}$.

Cor. 2. $r=\frac{2Ta-2p}{2pT+\overline{t-tt-2bt}\times a}$.

Cor. 3. *To find* T, put $b=0$, then in the equation $p+prt=\frac{tt-t}{2}ra+ta$, t will be found, the neareſt whole number. Then proceed as in Cor. 3. of the laſt Prop. to find b.

SCHOL.

In ſimple intereſt, each ſum of money is to gain intereſt after it becomes due. And compound intereſt, or intereſt upon intereſt, ſuppoſes that the intereſt itſelf ſhall alſo gain intereſt.

PROP.

PROP. IV.

Having given the principal, rate of intereſt, and time of continuance; to find the amount, at compound intereſt.

R = amount of 1*l.* in a year.
$1 : R :: R : R^2$, amount in 2 years.
$1 : R :: R^2 : R^3$, amount in 3 years.
$1 : R^b :: R^t : R^{t+b}$, the amount in $t+b$ years.
whence $1 : R^{t+b} :: p : pR^{t+b}$ the amount of p in the time $t+b$ or T. Therefore $pR^{t+b} = s$.

Cor. 1. $\overline{R^{t+b}-1} \times p =$ *the compound intereſt of* p *for the time* T.

Cor. 2. $p = \frac{s}{R^{t+b}}$.

Cor. 3. $\frac{\overline{R^{t+b}-1} \times s}{R^{t+b}} =$ *diſcount or rebate of* s *for the time* T.

For the diſcount is the difference between the ſum s, and its preſent worth, $= s - p = s - \frac{s}{R^{t+b}}$ $= \frac{R^{t+b}s - s}{R^{t+b}}$.

PROP. V.

Having given the annuity, time, and rate of intereſt; to find the money due at the end of that time, at compound intereſt.

By Prop. V. we ſhall have
$ba =$ rent due at the end of T or $t+b$ years without intereſt.
$aR^b =$ rent due at t years end, and its intereſt till T.
$aR^{t+b} =$ rent due at $t-1$ years end, and intereſt till T.

$aR^{2}+b$ = rent due at t—2 years end, and intereſt till T.
$aR^{3}+b$ = rent due at t—3 years end, and intereſt till T, and ſo on, till
$aR^{t-1}+b$ = rent due at 1 year's end, and intereſt till the end of T years.

Therefore $s = ba + R^{b} + R^{1+b} + R^{2+b} + R^{3+b} \ldots R^{t-1+b}$: × into a. $= Za + ba$, by ſubſtitution.

But by Cor. 3. Prop. XXVI. Geom. Proportion, the ſum of the ſeries $R^{b} + R^{1+b} + R^{2+b}$ &c. conſiſting of t terms $= \frac{R^{t}-1}{R-1} R^{b} = Z$: therefore

$$s = ba + Za = ba + \frac{R^{t}-1}{r} R^{b} \times a.$$

Cor. 1. $a = \frac{rs}{R^{t+b} - R^{b} + br}$.

Cor. 2. $R^{t+b} - R^{b} + bR - \frac{Rs}{a} = b - \frac{s}{a}$.

From which equation, R may be found, by extracting its root.

Cor. 3. *To find* T; put $b = o$, then $aR^{t} - Rs = a - s$, whence t will be found, the neareſt whole number. Then by the equation, Cor. 2. b will be found the ſame way.

PROP. VI.

Having given the annuity, time, and rate of intereſt; to find its preſent worth; at compound intereſt.

By Prop. IV. $s = pR^{t+b}$, and by Prop. V. $s = ba + \frac{R^{t+b} - R^{b}}{r} \times a$; therefore $p\,R^{t+b} = ba + \frac{R^{t+b} - R^{b}}{r} a$, then $prR^{t+b} = bra + \overline{R^{t+b} - R^{b}} \times a$; and $p = \frac{bra + R^{t+b}a - R^{b}a}{rR^{t+b}}$.

Cor.

Cor. 1. $a = \frac{prR^t+b}{br+R^t+b-R^b}$.

Cor. 2. $pR^{t+b+1} - pR^{t+b} = baR - ba + R^{t+b}a - R^ba$,
By which equation R may be found by extracting the root.

Cor. 3. *To find* T; put $b=0$, then $prR^t = R^ta - a$, or $R^t = \frac{a}{a-pr}$, whence t will be found a near whole number, which being had, b will be found, with a deal of trouble, by the equation $prR^t \times R^b = bra + \overline{R^t-1} \times R^ba$.

Cor. 4. *If t be infinite, then $pr = a$, and $p = \frac{a}{r}$, and $\frac{p}{a} = \frac{1}{r}$ = number of years purchase.*

For then $prR^{t+b} = R^{t+b}a$, as all the other quantities are infinitely less, whence $pr = a$.

These propositions and corollaries are best wrought by the help of logarithms; and $1+r$ may be put for R, as there is occasion, and the contrary; and also $t+b$ for T.

SCHOL.

If b be put $=0$, in these propositions; then will be had all the rules which are commonly laid down, by the writers upon interest.

These last propositions, suppose compound interest to be paid for parts of a year, which admit only of simple interest by law. Whence they are indeed no more than a mathematical speculation; as that method is not practised. Under a year compound interest converts to simple interest.

Fig.

ART. XI.

The Figures of Sines, Cosines, &c. erected upon all Points of the Arch; their Areas; and Solidities when erected upon the Surface of the Sphere.

PROP. I.

To find the area of the figure of versed sines, sines, cosines, tangents and secants.

HERE we suppose the arch of the circle, divided into an infinite number of equal parts, upon each of which is erected the sine, cosine, &c. belonging to that arch. To find the area of the curve formed thereby.

37. 1. *For the versed sines.* Let radius AC $= r$, arch AB $= z$, versed sine AP $= v$, sine BP $= y$, cosine CP $= x$, tangent AT $= t$, secant CT $= s$; then $2rv - vv = yy$, and $r\dot{v} - v\dot{v} = y\dot{y}$. Make $Ab =$ arch AB, and ordinate $bp = Ap = v$. Then the fluxion of the area $Abp = v\dot{z}$, but by the nature of the circle $\dot{z} = \frac{r\dot{v}}{y}$, and $v\dot{z} = \frac{rv\dot{v}}{y} = \frac{rv\dot{v}}{\sqrt{2rv - vv}}$ $= \frac{rv^{\frac{1}{2}}\dot{v}}{\sqrt{2r - v}}$. But by Form 10th, Fluxions, the fluent of $\frac{v^{-\frac{1}{2}}\dot{v}}{\sqrt{2r - v}} = \frac{\text{arch AB}}{r} = \varphi$. And (Form 11th) fluent of $\frac{v^{\frac{1}{2}}\dot{v}}{\sqrt{2r - v}} = r\varphi - \sqrt{2rv - vv} =$ arch AB$-y$; and fluent of $\frac{rv^{\frac{1}{2}}\dot{v}}{\sqrt{2r - v}}$ or $\frac{rv\dot{v}}{\sqrt{2rv - vv}}$ $= r$

$= r \times$ arch AB $-$ sine y, for the area of the curve Abp. Fig.

And when $Ab =$ quadrant AD, the area $= \frac{3.1416rr}{2} - rr$.

And when $Ab =$ a semi-circle, the area $= 3.1416rr =$ AFG.

2. *For the sines.* Make $Ab =$ arch AB (fig. 37.) 38.
and erect the ordinate $bp =$ sine BP, then the fluxion of the area $= y\dot{x} = r\dot{v}$, and the fluent is $= rv = r \times$ AP, for the figure of sines.

And when Ab is a quadrant, the area $= rr$.

And when Ab is a semi-circle, the area $= 2rr$.

3. *For the figure of cosines.* Make $Ab =$ arch 39.
AB, (Fig. 37.) and at b, erect $bp =$ CP the cosine; then the fluxion of the area is $= x\dot{z} = x \times \frac{r\dot{v}}{y} = \frac{-rx\dot{x}}{y} = \frac{-rx\dot{x}}{\sqrt{rr-xx}}$, and the fluent is $= r\sqrt{rr-xx} = ry$. When Ad is a quadrant, the area $= rr$, and there the curve ap interfects the axis at d, and forms the fame figure on the other fide. This Fig. is the fame as Fig. 38, beginning at the center, that is $dbp = Abp$.

When Ab is a femi-circle the whole figure $= 2rr$, as in Art. 2.

4. *For the figure of tangents.* Make $Ab =$ arch 40.
AB (Fig. 37.) and at b erect the ordinate $bt =$ the tangent AT. Then the fluxion of the area $= t\dot{z}$. But by Trigonometry, $x : y :: r : t = \frac{ry}{x}$; then $t\dot{z} = \frac{ry\dot{z}}{x} = \frac{ry}{x} \times \frac{r\dot{v}}{y} = \frac{rr\dot{v}}{x} = \frac{rr\dot{v}}{r-v} = \frac{-rr\dot{x}}{x}$, and the fluent $= rr \times \frac{2.30258}{-1} \times \log. \overline{r-v}$. But

when

Fig. 40. when the area $= o$, $v = o$, and the fluent corrected is $2.30258 \times \log. \frac{r}{r-v}$ for the area.

Or thus; fluent of $-\frac{rr\dot{x}}{x} = -rr \times 2.302 \log. x$; but when the area is o, $x = r$, therefore the correct fluent $= 2.30258 \times \log. \frac{r}{x}$.

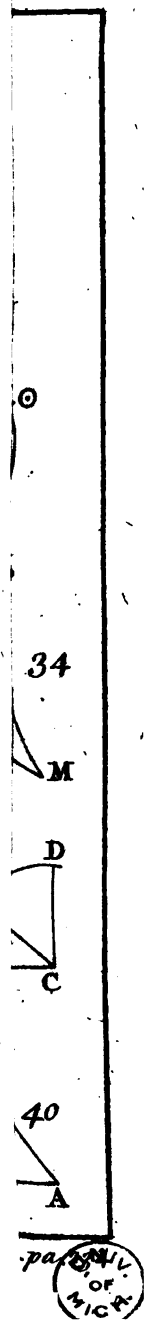
Otherwiſe thus.

Since $x : y :: r : t$, therefore $xx : yy :: rr : tt$, and $xx : xx+yy\ (rr) :: rr : rr+tt$. Whence $xx \times \overline{rr+tt} = r^4$, in Fluxions, $2x\dot{x} \times \overline{rr+tt}, + 2t\dot{t} \times xx = o$, and $t\dot{t} \times x = -\dot{x} \times \overline{rr+tt}$, therefore $\frac{\dot{t}t}{rr+tt} = \frac{-\dot{x}}{x}$. But $t\dot{z} = \frac{-rr\dot{x}}{x}$, therefore $t\dot{z} = \frac{rr\dot{t}t}{rr+tt}$ the fluxion of the area; and (Form 4th), the fluent is $2.30258\ rr \times \log. \frac{\overline{rr+tt}}{rr}$. = area A*bt*. Or $2.3025 \log. \frac{\overline{\int\int}}{rr} \times rr =$ that area.

Since the tangent of 90 degrees is infinite; if A*d* be a quadrant, the ordinate *dc* will be an aſymptote to the curve.

It is evident when A*d* is a quadrant, the area A*dc* is infinite.

41. 5. *For the figure of ſecants.* Make A*b* = arch AB, and ordinate *bp* = ſecant CT. Then the fluxion of the area $= \int\dot{z}$; and by Trigonometry $x : r :: r : f$, and $\int x = rr$, in fluxions $\int\dot{x} + x\dot{f} = o$, and $\dot{x} = \frac{-x\dot{f}}{f}$; and $\dot{z} = \frac{r\dot{v}}{y} = \frac{-r\dot{x}}{y} = \frac{-r}{y} \times$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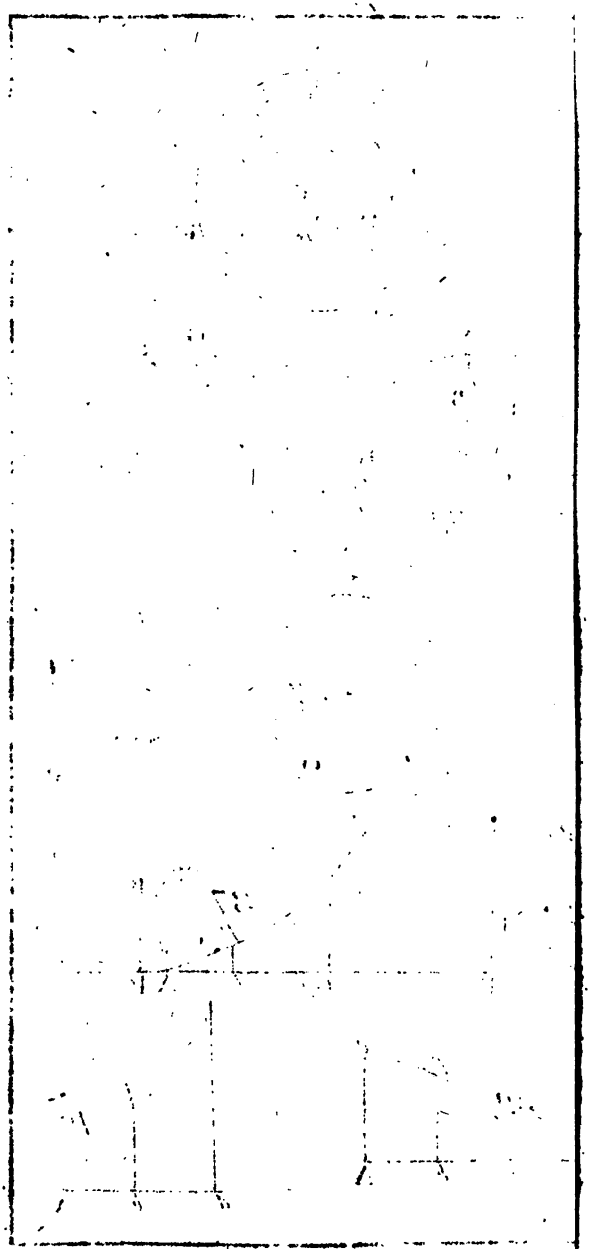

$\frac{-xf}{f} = \frac{rxf}{fy}$, and $f\dot{z} = \frac{x}{y} \times rf$. But $\frac{x}{y} = \frac{r}{t}$ Fig. 41.

$= \frac{r}{\sqrt{ff-rr}}$; whence $f\dot{z} = \frac{rrf}{\sqrt{ff-rr}}$. And (by Form 9th, Fluxions,) the fluent of $\frac{f}{\sqrt{-rr+ff}} =$ 2.30258 log. $\sqrt{ff} + \sqrt{-rr+ff}$. But when the area is $= 0$, $f = r$; therefore the fluent corrected is 2.30258 $\times$ log. $\frac{f+\sqrt{ff-rr}}{r}$; and the fluent of $\frac{rrf}{\sqrt{ff-rr}} =$ 2.30258$rr \times$ log. $\frac{f+\sqrt{ff-rr}}{r}$: for the area *Abpa*.

If *Ad* be a quadrant, the ordinate *dc* will be an asymptote to the curve; and the area *Adca* will be infinite.

From hence the meridional parts in Mercator's Chart may be calculated for any latitude AB or *Ab*. For the mer. parts for any latitude : is to the arch of latitude *Ab* : : as the sum of the secants : to the sum of as many radii : : or as the area *Aapb* : to *Ab* $\times$ radius *Aa*.

6. It is evident the area made by erecting the radii at all points of the arch AB is $= zr$ or $Ab \times r$,

PROP. II.

To find the sum of the squares of all the sines, cosines, versed sines, &c. erected upon every point of the arch AB; *or to find the solidities of the bodies formed thereby.*

1. *For the versed sines.* Denoting the quantities as in the first Prop. we shall have $vv\dot{z}$ for the fluxion 37

Fig. 37. fluxion of the ſolid; but $\dot{z} = \frac{r\dot{v}}{y} = \frac{r\dot{v}}{\sqrt{2rv - vv}}$,

whence $v v \dot{z} = \frac{r v v \dot{v}}{\sqrt{2rv - vv}} = \frac{r v^{\frac{3}{2}} \dot{v}}{\sqrt{2r - v}}$. But (by Form 10th,) fluent of $\frac{v - \frac{1}{2}\dot{v}}{\sqrt{2r - v}} = \frac{AB}{r}$, and fluent of $\frac{v^{\frac{3}{2}}\dot{v}}{\sqrt{2r - v}} = \frac{4rr}{1} \times \frac{\frac{3}{2} \cdot \frac{1}{2}}{2.1} \times \frac{AB}{r} + \frac{v^{\frac{3}{2}}\sqrt{2r - v}}{-2}$ $- \frac{\frac{3}{2} \cdot 2r}{-v} A = \frac{3}{2} r \times AB - \frac{v\sqrt{2rv - vv}}{2} + \frac{3r}{v} \times$ $\frac{-v}{2}\sqrt{2rv - vv}$, and the fluent of $\frac{r v^{\frac{3}{2}} \dot{v}}{\sqrt{2r - v}}$, or $\frac{r v^2 \dot{v}}{\sqrt{2rv - vv}} = \frac{3}{2} rr \times AB - \frac{3rr + rv}{2}\sqrt{2rv - vv}$, for the ſolid content of all the verſed ſines.

When AB is a quadrant, $v = r$, and the ſolidity $= 3r^3 \times \frac{3.1416}{4} - 2r^3$.

And when AB is a ſemi-circle, the ſolidity $= \frac{3}{4} r^3 \times 3.1416$.

38. 2. *For the ſines.* The fluxion of the ſolid $= yy\dot{z}$, but by the nature of the circle $\dot{z} = \frac{r\dot{y}}{x} = \frac{r\dot{y}}{\sqrt{rr - yy}}$

whence $yy\dot{z} = \frac{ry^2\dot{y}}{\sqrt{rr - yy}}$. But (Form 10th.) fluent of $\frac{\dot{y}}{\sqrt{rr - yy}} = \frac{z}{r}$, and (Form 11th.) fluent of $\frac{y^2\dot{y}}{\sqrt{rr - yy}} = \frac{rz}{2} - \frac{y\sqrt{rr - yy}}{2}$, and fluent of $\frac{ry^2\dot{y}}{\sqrt{rr - yy}} = \frac{rz - xy}{2} r =$ area APB $\times r$. (Fig. 37.) for the ſolid of ſines.

When

When Ab is a quadrant, the ſolidity = quadrant ACD $\times r = .7854r^3$. Fig. 38.

When Ab is a ſemi-circle, the ſolidity = $\frac{3.1416}{2} r^3$.

3. *For the coſines.* The fluxion of the ſolid is $xx\dot{z} = xx \times \frac{r\dot{y}}{x} = rx\dot{y} = r\dot{y}\sqrt{rr-yy}$; the fluent of $\frac{\dot{y}}{\sqrt{rr-yy}} = \frac{\text{arch AB}}{r}$, by Form 10th. And fluent of $\dot{y}\sqrt{rr-yy} = \frac{2\times\frac{1}{2}rr}{2}\varphi + \frac{y\sqrt{rr-yy}}{2}$; and (Form 13.) fluent of $r\dot{y}\sqrt{rr-yy} = \frac{r\times\text{AB}+xy}{2}$ $r = \frac{rz+xy}{2} r$, for the ſolid of coſines (Fig. 37.) 39.

When Ad is a quadrant, the ſolidity = $\frac{rr\times\text{AD}}{2}$ = quadrant AD $\times r$.

4. *For the ſolid of tangents.* Here the fluxion of the ſolid is $tt\dot{z} = \frac{ttr\dot{v}}{y} = \frac{-ttr\dot{x}}{y}$. But by Art. 4th of the laſt Prop. $-\dot{x} = \frac{\dot{x}tt}{rr+tt}$, and $\frac{x}{y} = \frac{r}{t}$, therefore $tt\dot{z} = \frac{-ttr\dot{x}}{y} = \frac{ttr}{y} \times \frac{\dot{x}tt}{rr+tt} = \frac{x}{y} \times \frac{rt^3\dot{t}}{rr+tt} = \frac{r}{t} \times \frac{rt^3\dot{t}}{rr+tt} = \frac{rrtt\dot{t}}{rr+tt}$. But (by Form 5th,) the fluent of $\frac{\dot{t}}{rr+tt} = \frac{\text{AB}}{rr}$; and (Form 11th,) the fluent of $\frac{t^2\dot{t}}{rr+tt} = \frac{-rr}{1}\varphi + \frac{1.t}{1} = -\text{AB}+$ $t =$ 40.

Fig.

40. $t = t - z$, and the fluent of $\frac{rr\dot{t}^2\dot{t}}{rr+tt} = \overline{t-z} \times rr$ $= \overline{AT - AB} \times rr$, for the ſolid of tangents (Fig. 37.)

When A*d* is a quadrant, the ſolidity is infinite, becauſe the tangent is infinite.

41. 5. *For the ſecants.* The fluxion of the ſolid is $\dddot{z} = \frac{-\dddot{r}\dot{x}}{y}$. But by the 5th Art. laſt Prop. $-\dot{x} = \frac{x\dot{f}}{f}$, and $\frac{x}{y} = \frac{r}{\sqrt{ff-rr}}$. Wherefore $\dddot{z} = \frac{-r\ddot{f}\dot{x}}{y} = \frac{r\ddot{f}}{y} \times \frac{x\dot{f}}{f} = \frac{x}{y} \times r\ddot{f} = \frac{rr\ddot{f}}{\sqrt{ff-rr}}$, and the fluent is $rr\sqrt{ff-rr} = rrt$, for the ſolid of ſecants.

It is plain when the arch AB or A*b*, that is A*d*, is a quadrant, the ſolid is infinite.

6. *For the ſum of all the ſquares of the radii erected on the arch* AB, we have $rr\dot{z}$ for the fluxion of the ſolid; and the fluent is rrz for the ſum of all the radii ſquare, or the ſolid formed thereby.

Cor. 1. *It follows from the three laſt Articles, that the ſolid of ſecants is equal to theſe two ſolids, the ſolid of tangents, and the ſolid formed of all the radii.*

Cor. 2. *Hence alſo the ſum of the ſquares of all the fines in the quadrant = half the ſum of the ſquares of as many radii.* From Art. 2. and 6.

PROP. III.

To find the ſum of all the fines, coſines, verſed fines, &c. erected upon every point of the ſpherical ſurface; or to find the ſolids made thereby.

42. Here we ſuppoſe the circle ABD to revolve about the axis AC, and to deſcribe a ſpherical ſurface. And

And that upon the annulus described by BP, there is erected at every point of it, perpendiculars equal to BP for the sines, or CP for the cosines, &c. and so for every annulus. But this is to be understood, that every annulus is laid down in a plain, upon an abscissa equal to the arch AB; then each annulus will be the base upon which we are to erect the said perpendiculars. Then these form so many solid bodies whose contents we must investigate. Fig. 42.

1. *For the versed sines.* Supposing the quantities denoted as in Prop. I. and let $c = 3.1416$, then $2cy\dot{z}$ is the area of the annulus at B; multiply this by the versed sine v, and we have $2cvy\dot{z} =$ fluxion of the solid. But by nature of the circle $y\dot{z} = r\dot{v}$, therefore the fluxion of the solid $= 2crv\dot{v}$, and the fluent $= crvv$, for the solid formed by all the versed sines standing on the spherical surface described by APB.

When the arch AB becomes the quadrant AD, then the solid $= cr^3$: for then $v = r$.

And when the arch becomes a semi-circle, or the surface, that of the whole sphere, the solidity $= 4cr^3$. For then $c = 2r$.

2. *For the sines.* The general area of the annulus BP is $2cy\dot{z}$; which multiplied by the sine y, is $2cyy\dot{z}$; expunge $y\dot{z}$, by taking its equal $r\dot{v}$; then the fluxion of the solid is $2cry\dot{v}$. But the fluent of $y\dot{v}$ = area BAP. Therefore the fluent of $2cry\dot{v}$ $= 2cr \times$ area ABP, for the solidity.

And for the hemisphere, the solidity $= 2cr \times$ quadrant ADC $= \frac{ccr^3}{2}$, for the solid of all the sines, upon the hemisphere.

And consequently the sum of all the sines upon the whole sphere $= ccr^3$.

3. *For*

Fig. 42. 3. *For the cosines.* The area of the annulus BP is $2cy\dot{z}$, which multiplied by the cosine x is $2cyx\dot{z}$, put $-r\dot{x} = y\dot{z}$, and the fluxion of the solid is $-2crx\dot{x}$, and the fluent is $-crxx$. But at A the solid $= 0$, when $x = r$, therefore the fluent corrected is $cr^3 - crxx = cr \times rr - xx = cryy$, for the solidity. And hence the sum of all the cosines upon the hemisphere $= cr^3$.

4. *For the sum of all the tangents.* The area of the annulus BP $= 2cy\dot{z}$, which multiply by the tangent t, then $2cyt\dot{z}$ = fluxion of the solid, but $\dot{z} = \frac{r\dot{y}}{x}$; therefore the fluxion of the solid $= \frac{2crty\dot{y}}{x} = \frac{2crty\dot{y}}{\sqrt{rr-yy}}$. But by Trigonometry x $(\sqrt{rr-yy}) : y :: r : t = \frac{ry}{\sqrt{rr-yy}}$, therefore the fluxion of the solid $= \frac{2cry\dot{y}}{\sqrt{rr-yy}} \times \frac{ry}{\sqrt{rr-yy}} = \frac{2crry^2\dot{y}}{rr-yy} = 2cr^4 \times \frac{\dot{y}}{rr-yy} - 2cr\dot{y}$. But (Form 6,) the fluent of $\frac{\dot{y}}{rr-yy} = \frac{2.30258}{2r} \log. \frac{r+y}{r-y}$. Therefore the fluent $= 2.30258\, cr^3 \times \log. \frac{r+y}{r-y} : -2cry$, the solid content of all the tangents.

And in the hemisphere, the sum of all the tangents is infinite.

5. *For the sum of the secants.* Here $2cy\dot{z}$ = area of the annulus, and by Trigonometry $x : r :: r : f$, and $fx = rr$, in fluxions $f\dot{x} + x\dot{f} = 0$, and $\dot{x} = \frac{-x\dot{f}}{f}$; but $\dot{z} = \frac{-r\dot{x}}{y}$; whence the fluxion of the solid

folid $= 2cy\dot{f}\dot{z} = -2cf\dot{r}\dot{x} = -2cf\dot{r} \times \frac{-x\dot{f}}{f} =$ Fig. 42.
$2cr\dot{x}f = 2crf \times \frac{r\dot{r}}{f} = \frac{2cr^3 \times \dot{f}}{f}$, and the fluent $=$ $2cr^3 \times 2.3025$ log. f, for the folid. But when the folid is o, $f = r$; therefore the fluent corrected $=$ $2cr^3 \times 2.30258$ log. $\frac{f}{r}$, for the ſum of all the ſecants on the ſurface ABP.

And in the hemiſphere, the ſum of all the ſecants is infinite.

6. *For the ſum of all the radii.* The annulus $=$ $2cy\dot{z} = 2cr\dot{x}$. And the fluxion of the ſolid $=$ $2cr^2\dot{x}$. And the fluent is $2crrx$, for the ſum of the radii upon the ſurface deſcribed by BAP.

And the ſum of all the radii upon the hemiſphere, is $2cr^3$; and upon the whole ſphere $4cr^3$.

Cor. 1. *The ſum of all the verſed ſines on the hemiſphere is equal to all the coſines on the hemiſphere.*

Cor. 2. *The ſum of the ſines, coſines, and radii, upon the hemiſphere, are as* $\frac{1}{2}c$, 1 *and* 2.

For they are as $\frac{ccr^3}{2}$, cr^3, and $2cr^3$.

Fig.

ART. XII.

FORTIFICATION.

FORTIFICATION is an art that teaches how to fortify any place, city, or town, by incloſing it with walls and other works to make it ſecure againſt the attack of an enemy; where a ſmall number of men may defend themſelves againſt an army. It is called *regular fortification* when it is built upon a regular poligon, as a pentagon, hexagon, heptagon, &c. and *irregular fortification*, when the figure is irregular, the parts not uniform, the angles and ſides unequal.

DEFINITIONS.

DEF. I.

43. The *poligon* is the figure upon which the fortification is built; the *exterior poligon* is the figure of the outſide, and the *interior poligon* is the figure on the inſide. Thus ABOPQ is the exterior poligon, and GKLMN is the interior poligon, GK the ſide of it.

DEF. II.

A baſtion is a great work or bulwark built before an angle of the poligon, ſtanding out towards the field, as FDAEH. It is built of ſtone or brick, or ſometimes of earth. There are ſeveral ſorts of theſe, as a *full baſtion* is one quite filled up with earth. A *hollow baſtion* is that wherein an empty ſpace is left, to hold magazines, proviſions, &c. A *flat baſtion* or *plat baſtion*, is that made upon a right line and not at an angle. *Irregular baſtion*, one

one whoſe parts are out of due proportion. A *cut baſtion* is one with two points. A *detached baſtion*, one ſeparated from the body of the other works. A *demi-baſtion* is but half a baſtion, as ADFG. A *double baſtion* is one built upon another; or one with another. Fig. 43.

DEF. III.

The *demi-gorge* or *gorge line*, is a part of the ſide of the interior poligon, from the outſide to the center of the baſtion, as FG.

DEF. IV.

The *face* is the outſide of the baſtion facing the field, as DA.

DEF. V.

The flank is that part of the baſtion joining to the face, as DF. It is called a *right flank* when perpendicular to the curtain. An *oblique flank* is one that makes oblique angles with the curtain. A *retired* or *covered flank* is one made hollow. The *ſhoulder* of the flank is the point D. A *ſecond flank* takes in a part of the curtain.

DEF. VI.

The *capital* is a part of the radius of the exterior poligon, contained between the angle of the interior poligon, and the extreme point of the baſtion, as GA.

DEF. VII.

The *curtain* is that part of a ſide of the interior poligon, that lies between two baſtions, as HI.

DEF. VIII.

The *line of defence* is the line drawn from the end of the flank to the extreme point of the baſtion, as AI.

Fig. 43.

DEF. IX.

The *angle at the center*, is the angle that a ſide of the poligon ſubtends, as GCK.

DEF. X.

The *angle of the poligon* is the angle made by two ſides of the poligon, as NGK. And the *angle* of the *triangle* is CGK half the angle of the poligon.

DEF. XI.

The *angle of the baſtion*, is the angle made by the two faces of the baſtion, as DAE. Some call it the flanked angle.

DEF. XII.

The *angle of the ſhoulder*, is that made by the face and the flank, as ADF.

DEF. XIII.

The *angle of the flank*, is that made by the flank and the curtain, as EHI.

DEF. XIV.

The *angle of tenail*, is that made by two lines of defence, as ATB, called alſo the *flanking angle*.

DEF. XV.

Angle diminiſhed, is the angle which the flank ſubtends at the oppoſite end of the curtain, as EIH.

DEF. XVI.

A line raiſant is a line running in direction of another line, as the face of the baſtion.

DEF. XVII.

A *re entrant angle*, is that which has its point inwards; it is alſo called a *tenaile angle*. And a *ſa-*

liant

liant angle is one that looks, or points with its legs, inwards, or has the angular point outwards, contrary to the entrant angle. Fig. 43.

Beſides GCH is the *flank-forming angle*; and GI the *lengthened curtain.*

General Rules and Maxims in FORTIFICATION.

I.

Every part of a fortified place ought to be defended from another place, and therefore ought to lie open to its view. For if any part of the fortification is not flanked or defended, the enemy could lodge there and be covered; and ſo the place would be more eaſily taken.

II.

The line of defence, or which is nearly the ſame thing, the ſide of the interior poligon, ought to be within muſket ſhot; and therefore ought to be about 700 feet, or between 500 and 800 feet. For though both cannons and muſkets are uſed, yet the length ought to be adapted to the reach of a muſket; becauſe a cannon requires a deal of time and trouble to charge and diſcharge; and conſumes a deal of powder; often makes random ſhots; and is liable to be diſmounted by the enemy's battery; ſo that muſkets muſt be provided for.

III.

The line of defence muſt be a line raiſant, when there is no ſecond flank; and muſt terminate in the angle of the flank. When the face of the baſtion continued cuts the curtain, the ſpace between this point of interſection and the angle of the flank, is that which is called the *ſecond flank*. And the line of defence muſt never interſect the flank, for then

Fig. there would be a part of the flank, from which the face of the baſtion could not be ſeen; the baſtion ought to be defended from the whole flank.

IV.

The flanks, demi-gorges and ſecond flanks, ought to be made as great as poſſible, provided the reſt of the works are not injured thereby. For then they are fitter for defending a place, and can hold more ſoldiers, and more cannon.

V.

The angle of the baſtion ought to be about 70 or 80 degrees, not ſo well when 90. Here regard is to be had to the number of baſtions or ſides of the poligon. For a leſs number of baſtions requires a more acute angle; and a greater number a more obtuſe one. When the angle of the baſtion is too acute, it is ſo much the ſooner battered to pieces by the enemy. The baſtion itſelf ought to be of a middling ſize, neither too great nor too little, great ones are too expenſive, and ſmall ones want room and convenience. The face ought never to be leſs than half the curtain.

VI.

It is better to have a ſecond flank than otherwiſe. For by this means more room will be gained for fire. Alſo by this one may ſhoot further into a breach made in the baſtion, and hinder the enemy from lodging there; and one may do more hurt to the enemy paſſing the ditch, as more of the ditch can be diſcovered.

VII.

The flank ought to be perpendicular to the raiſant line, or rather ſomething greater than a right angle, otherwiſe it ought to be drawn to the center of

of the place. Some draw it perpendicular to the face, but this makes the gorge too narrow. Others make it perpendicular to the curtain, but this is attended with the ſame inconvenience. And both theſe methods make the flank too oblique. But if the angle be much greater than a right angle, the flank is too much expoſed to the enemy's cannon. Fig.

VIII.

The greater the angle between the ſide of the outward poligon and the face, the better is the face defended; for it is better ſeen from the flank; and beſides, a ſecond flank may be obtained by this means.

IX.

A part of the flank ought to be covered. As the flank is the chief part that defends the place, and fights for its ſafety; every thing that can, ought to be done for its ſtrength and preſervation, and to hinder it from being demoliſhed. Therefore an *orillon* is very uſeful to cover the flank. An orillon is an additional part of the baſtion, being built farther out at the ſhoulder, by which means the flank will be hollow, and is better ſecured from the enemy's ſhot.

X.

A fort ought to be equally ſtrong throughout; and be ſo high as to command all places round about. And all outworks muſt be lower than the body of the place. If a fort be weaker in one place than another, the enemy will certainly attack it in that place. If any place be out of ſight of the fort, the enemy could cover his deſigns, and carry on his approaches unnoticed.

XI.

The works that are moſt remote from the center of the place, ought to be open to thoſe that are

Fig. nearer. And therefore thoſe that are neareſt the center ought to be higher than thoſe that are further off. For thoſe that are furtheſt off, are moſt liable to be taken by the enemy; but lying open to the fort they may more eaſily be defended, and the enemy hindered from covering himſelf therein.

XII.

The more acute the angle at the center is, the ſtronger the place is; for then there will be more ſides of the poligon.

XIII.

All theſe maxims muſt be ſo practiſed, as to conſpire and agree with one another, as far as poſſible. Yet it often happens when we adhere too ſtrictly to one maxim, we deviate conſiderably from another. Thus, if we increaſe the ſecond flank, either the oppoſite face of the baſtion muſt be increaſed, or the flank diminiſhed; if the angle of the baſtion be made very great, the defence it has from the flank or ſecond flank is diminiſhed, and it is more expoſed to the enemy's cannon. If the gorge be made great, it cauſes the face of the baſtion to be too great. There is always advantage attended with diſadvantage, and our reaſon muſt judge how far we may ſtretch one maxim, and not encroach too much upon another; but endeavour to make them all agree as near as can be.

Likewiſe the expence of the works is a material article, and ought to be well conſidered; leſt more money be laid out upon the defence of a place, than all the advantages of it amount to.

PROB.

PROB. I.

To deſcribe a regular fortification about any place given.

Fig.

It very ſeldom happens that a place is regular of itſelf, yet when there is field room enough about it, one may make it a regular one.

In the firſt place fix the ſide of the figure, as ſuppoſe 700 feet, then divide the circumference of the place by that number, and the quotient gives the number of baſtions. The quotient will ſeldom come out a whole number, therefore make the ſide ſomething more or leſs than 700, that the quotient may come out an integer. Suppoſe then there are 5 ſides, or that the figure is a regular pentagon. The angle at the center GCK is = 72°, and the half is 36. Then S.36 : ½GK (350) : : rad. : CG the radius of the circumſcribing circle 595. Therefore with the radius 595 deſcribe a circle from the center C. And in this circle inſcribe the pentagon GKLMN, each ſide being 700 feet. Produce CG, ſo that GA may be 194 feet, or thereabouts, for the capital. From G ſet off GF and GH = 146 for the demi-gorge. Through F and H draw CFD, CHE, and ſet off FD, HE = 117 feet, for the flank. Then draw AD and AE, for the face. Or elſe ſet off the demi-gorges NS and KI as before; then through D and E draw SDA and IEA to interſect in A, the angle of the baſtion. Or the flanks EH, DF may be otherwiſe drawn, to make a certain angle with the curtain HI and FS; or with the raiſant lines IA, SA. And thus if at each of the other angles K, L, M, N, a baſtion be drawn; then the fortification will be finiſhed, when the counterſcarp or outſide of the ditch is drawn, *abcdef*, &c. being parallel to the faces of the baſtions, 20 fathom wide. 43.

If

Fig. 43. If inſtead of drawing AD, AE to S and I, they be drawn to ſome point of the curtain, then you will have a ſecond flank.

Note, all theſe things may be laid down on the ground as well as on paper, by the help of the Propoſitions in Sect. I. of the Surveying. And to ſave the labour of making a deal of calculations, I have here inſerted (from Mr. *Ozanam*) a Table containing all the lines and angles of a fortified place, from the ſquare to the dodecagon, ſuppoſing the ſide of the interior poligon 120 fathom. As for triangles, they are hardly to be fortified any other way but with half baſtions.

poligons	IV.	V.	VI.	VII.	VIII.	IX.	X.	XI.	XII.
angle at the center	90 0	72 0	60 0	51 26	45 0	40 0	36 0	32 44	30 0
angle of the poligon	90 0	108 0	120 0	128 34	135 0	140 0	144 0	147 16	150 0
angle of the flank	120 58	112 59	108 7	105 7	102 28	100 39	99 16	98 32	97 38
angle of the ſhould.	130 40	126 18	124 56	125 21	126 14	127 55	130 0	129 21	128 54
angle of baſtion	70 36	81 22	86 22	88 6	87 28	85 28	82 34	85 38	87 28
demi-gorge	24 0	25	26	27	28	29	30	30	30
curtain	72 0	70	68	66	64	62	60	60	60
flank	16	20	24	28	32	36	40	40	40
line of defence	F *f* 117 3	*f* 117 : 5	119.	*f* 120 : 3	*f* 122 : 5	126	129	127	*f* 125 : 4
face	*f* 36 1	*f* 37 : 5	*f* 40 : 1	*f* 42 : 2	45 : 2	48 : 5	52 : 3	50 : 0	49 : 1
capital	28.	33 : 3	39 : 4	46 : 1	53 : 4	61 : 2	69 : 4	67 : 5	67 : 3
radius	84 : 5	102 : 1	120	138 : 2	156 : 5	175 : 2	194 : 1	217 : 5	231 : 5

The Table above will be ready for laying down the fortification of any place that has no more than 12 ſides. And the computation of it is as follows, for any of the figures, as ſuppoſe a pentagon.

1. The

1. The side of the poligon GK being 120 fa- Fig. thoms, the demi-gorge FG is 25 for the pentagon, 43. the flank FD = 20.

From the side of the poligon GK	120
subtract twice the demi-gorge GH	50
2. Remains for the curtain	70

3. If 360 be divided by 5 the number of sides, the quotient is 72 the angle at the center GCK.

And 72 taken from 180, leaves 108, half of which 54 is,

3. The angle at the base CGK, or half the angle of the poligon.

4. To find the radius CG, by Trigonometry,

As cosine of CGK	54°	9.76921
to half GK	60	1.77815
rad.		10.
CG	102.1	2.00894

5. To find the angle of the flank DFS or CFG, CG + GF = 127.1, and CG — GF = 77.1, and 180 — FGC (54) = 126 = sum of the angles F and C, the half is 63.

Then sum sides	127.1	2.10401
difference	77.1	1.88705
tan. $\frac{F+C}{2}$ (63)		10.29283
		12.17988
tan. $\frac{F-C}{2}$,	49 59	10.07587

then 63 + 49 59 = 112 59 = angle DFS of the flank,

and 63 — 49 59 = 13 : 1 = angle GCF.

6. To find the angle of the shoulder ADF or SDF, SF + FD = 90, and SF — FD = 50, and 180 — DFG (113) = 67, half is 33 30 = $\frac{D+S}{2}$.

Sum

Fig. 43.

Sum sides	90		1.95424
diff. sides	50		1.69897
tan.	33	30	9.82078
			11.51975
tan. $\frac{D-S}{2}$,	20	12	9.56551, and

33 30 + 20 12 = 53 42 = FDS.

Then 180 — 53 42 = 126 18 = angle ADF of the ſhoulder.

Alſo 33 30 — 20 12 = 13 18 = angle DSF,

7. To find the angle of the baſtion EAD. To ADC (126 18) add DCA or FCG (13 1), the ſum is 139 19, which taken from 180°, leaves 40 41 for DAC half the angle of the baſtion; therefore the angle of the baſtion is 81 22.

8. To find the line of defence SA, and capital AG. SG = SF + FG = 95.

Then S.SAG (40 41)	9.81416
GS (95)	1.97772
S.AGF or SGC (54 00)	9.90795
	11.88567
SA 117.9	2.07151
S.ASG *or* DSF (13 18)	9.36182
	11 33954
GA 33.5	1.52538

So the line of defence is 117^{F} 5^{f}, and the capital 33^{F} 3^{f}.

9. To find the face AD. CA = CG + GA = 135.6.

Then S.ADC *or* SDF (53 42)	9.90629
AC (135.6)	2.13225
S.ACD (13 1)	9.35263
	11.48488
AD 37.9	1.57859

So the face is 37 fathoms, 5 feet.

And after the ſame manner, the requiſites may be calculated for any figure or poligon.

To

To conſtruct the figure from the Table. Fig. 43.

The ſide of the poligon being 120 fathom, the radius for the pentagon is $102^F : 1^f$; with this radius, and center C deſcribe a circle, in which inſcribe the pentagon GKLMN. Through the points G, K, L, M, N, draw the lines CA, CB, CO, CP, CQ. And ſet off the capital 33 fathom, 3 feet, from theſe points = GA, KB, LO, MP, NQ. Then the demi-gorge being 25; make GF, GH, NS, KI, &c. each = 25. Through F, H, &c. draw the lines CD, CE, &c. And the flank being 20, make FD, HE, &c. = 20. Then from S, I, &c. draw lines through D, E, &c. to interſect in A, &c. making DAE the angle of the baſtion, and ſo of the reſt, till all the parts be finiſhed.

Note, if you would have the ſide of the poligon more or leſs than 120 fathom, it is but taking a length proportionally bigger or leſs than a fathom, and dividing it into 6 parts. Then all the numbers may be taken out of the Table the ſame as before, and ſet off, after this new meaſure or ſcale, either upon paper, or on the ground.

SCHOL.

We have laid down in this Prob. a general way of drawing a fortification, as is commonly practiſed. But different engineers have different ways and meaſures, for every article here mentioned. I ſhall therefore mention the ſeveral methods that different writers have laid down for that purpoſe.

1. In the method before laid down, the faces DA, EA of the baſtion are formed by drawing lines from the angle of the flank, SA and IA. But ſome chooſe rather to make the angle DAE a right angle, by which means they will generally have a ſecond flank, if the poligon has many ſides. Now

if

Fig. 43. if DAE = 90°, its half is 45, which taken from SGC, half the angle of the poligon, leaves the diminished angle ASG, to which add the angle of the flank DFS, gives the angle of the shoulder ADF. The second flank is found by producing AD to cut the curtain.

2. Others make the gorge-line FH equal to the capital GA. Here the demi-gorges FG, GH being given, and the angle of the poligon FGH; the right line FH will be found by plain Trigonometry; then CG + FH (AG) = CA. Then the angle of the flank DFS or GFC will be known by the Table; and in the triangle CFG, all the angles being given, and the sides FG, GC; the side CF will be found. To this add the flank FD (taken as you please), gives CD. In the triangle ACD, the sides AC, DC, and angle ACD (from the Table) being given; the side DA and angle DAC will be found; and 2DAC = DAE the angle of the bastion.

3. Others construct the Problem after this manner. Let P be = the length of the perpendicular drawn from the center C to the middle of any side of the poligon, n = number of sides of the poligon. Then make the demi-gorge GH or GF = $\frac{2}{n+1}$ P, and the capital AG = $\frac{3}{n+1}$ P. Then draw the raisant lines SA, IA; and through F and H, draw from the center C, the flanks FD, HE. This is a good construction. If you have a mind to compute the quantities by Trigonometry, you will have this proportion,

radius,
tan. half the angle of the poligon KGC,
so half the side GK,
to the perpendicular P.

Then we shall have FG or GH, and AG. Then GK — GH = HK or GI. Then in the triangle AGI, there is given AG, GI, and the angle AGI;

to

to find the angle GIA, and the angle GAI, or half the angle of the baſtion. The radius CG is found by this proportion. Fig. 43.

As the ſine of half the angle of the poligon CGK,
to the perpendicular P,
ſo is radius,
to CG.

Then AC is known. And in the triangle CGH, there is given CG, and GH, and the angle CGH; to find the angle CHG or EHI, the angle of the flank, and the ſide CH, and the angle GCH or ACE, then the angle of the ſhoulder AEC is known. And in the triangle CAE, the ſide CA, and all the angles are given; to find the ſide AE, the face. Laſtly, in the triangle IEH, all the angles are given, and the ſide HI; to find the flank HE.

It may be obſerved, that this method makes no ſecond flanks.

4. Another way of fortifying is this. Let AB be the outward ſide of the poligon. Make the angles CAI, CBH, each 45 degrees. Biſſect the angle CBH by the line BI, to cut AI in I; and draw IH parallel to AB, to cut BH in H; then HI is the curtain. Then draw the flank HE perpendicular to AI, and ſo of the reſt; ſo the ſhoulder will be a right angle. This is to be performed from the ſquare to the octagon. But in poligons of more ſides, the flank HE muſt be perpendicular to the curtain IH. And all the requiſites may be computed trigonometrically like the reſt. But enough has been ſaid about ſuch computations.

5. This is another method of fortifying. Make the exterior ſide BO = 180 fathoms, to the middle of BO draw the perpendicular CR. Make VR = 30 fathoms (but in the ſquare only 24); draw through V, the lines of defence OX, BY. Make the faces B1, O2, each = 55 fathoms. Then draw

Fig. 43. draw the flanks 1X, 2Y, perpendicular to the raiſant lines XO, YB. And through the points of interſection X, Y, draw the curtain XY.

6. Another way of fortifying. Make the inner ſide of the poligon GK = 120 fathom; make GH and IK = a fifth part of GK, for the demi-gorges; and FD, HE = a fourth part of GK, for the flanks. And make the angle of the flank SFD or IHE = 100 degrees. Through D and E draw the raiſant lines SDA and IEA interſecting in A, the angle of the baſtion, and the faces are DA and EA.

7. Or thus, if the inward ſide GK be 140 fathoms, the demi-gorge GH, IK muſt be 28; the curtain HI = 84, and the flank EH = 35; then make the angle of the flank EHI = 100 degrees. Then draw through E, the line of defence IEA; likewiſe draw the line of defence SDA, to interſect IA in A, the angle of the baſtion, whoſe faces are AD, AE.

8. Otherwiſe thus; let n = number of ſides of the poligon; then make the diminiſhed angle AIH $= 45 - \frac{120}{n}$ degrees = angle ASF. Then the lines IA and SA being drawn, the interſection gives the angle of the baſtion at A. Or rather make the angle ABH, and BAI $= 45 - \frac{120}{n}$. Make BH, AI each = $\frac{7}{10}$ of AB. Draw HI for the curtain. Let AE = ET = AD, then AE, AD, will be the faces of the baſtion. And draw EH for the flank, and ſo of the reſt.

9. Or thus; upon the outward ſide BO let fall the perpendicular CR, which biſſects BO; take RV = $\frac{1}{8}$BO for the ſquare, $\frac{1}{7}$BO for the pentagon, or $\frac{1}{6}$BO for the hexagon, heptagon, &c. through V draw the lines of defence OVX, BVY. Make O2, and B1, each = $\frac{1}{7}$BO. And make 1Y, 2X each

each = 12, a line drawn from 1 to 2. Then draw XY for the curtain; and 1X, 2Y, for the flanks. This is a very good method. Fig. 43.

10. Or thus; make the inner ſide GK 130 fathom; make the demi-gorges GF, GH, each = 25, and likewiſe the flanks DF, EH = 25; and draw the flanks perpendicular to the curtain HI, FS. And make the ſecond flank 3 fathom. From which point draw a line through E or D, will interſect the radius CG produced, in A the point of the baſtion. But the ſquare or pentagon are better without ſecond flanks.

Or one may make the flanks and demi-gorges a ſixth part of GK. And make the angle of the baſtion DAE a right angle, eſpecially in the hexagon, heptagon, &c. whence the capital will be equal to the gorge line.

There are ſeveral other methods of fortifying; and in all theſe methods each of them has ſome advantages, which the others have not; but which, to chooſe, muſt be left to the judgment of the engineer, who muſt take that method which beſt ſuits his particular purpoſe.

PROB. II.

To conſtruct an irregular fortification about a place.

Many places cannot be incloſed by a regular figure or a regular fortification; in which caſe all that can be done is to reduce it as near to a regular form as is poſſible; and then we muſt apply the rules of a regular fortification, as far as the nature of the place will admit of. The principal deſign both in regular and irregular fortification is to make all the parts equally ſtrong. For if ſome be weaker than others, they will certainly be attacked in the weakeſt places. Longer ſides of a poligon

Fig. 44. are ſtronger than ſhorter ſides, provided they be within muſket ſhot. Alſo greater angles of a poligon are ſtronger than leſs or more acute angles. But to come to particulars.

1. Let GKLMN be an irregular poligon; find the center C of a circle whoſe circumference paſſes through three of the moſt diſtant points K, M, N. Draw lines from the center C to all the angles of the poligon, CK, CG, CN, &c. and meaſure all the angles at the center KCG, GCN, &c. and compare them with the angles at the center in the general Table, to find which they come neareſt to. Then fortify that ſide by the rules of that poligon, that is, divide that ſide into 120 equal parts, which will ſerve for a ſcale; then ſet off the gorge and flank, by the numbers in the Table belonging to that poligon. As if the angle GCK be 46 degrees, 360 divided by 46, give almoſt 8 for the quotient, therefore GK muſt be fortified as an octagon. And then the demi-gorge will be 28, the flank 32, which muſt be drawn from the center C. After the ſame manner muſt the demi-gorges and flanks be made for the reſt of the ſides, and the baſtions drawn.

2. Or thus; when the poligon is very irregular, take two adjoining ſides GK, GN; and find the center of a circle O, paſſing through the three points K, G, N. Then meaſure the angles KOG, GON. Then the flanks and demi-gorges are to be taken from the Table as before, according to the poligons they agree to, and the half baſtions for each ſide compleated. When the poligon has an odd ſide; biſſect the two angles at the ends of it, by two right lines, whoſe interſection will give the center required; then make two half baſtions as before: a good method.

3. Otherwiſe thus; biſſect all the angles of the irregular poligon by right lines. The two lines drawn from the two ends of any one ſide, will, by their

their interſection, give the center of a ſuppoſed re- Fig.
gular poligon, one ſide of which poligon is the in- 44.
cluded ſide. And that ſide is to be fortified accord-
ing to the angle at the center. Thus if the angles G, K, be biſſected by the lines GC, KC; then C will be the center of the poligon, and GK a ſide of it. And the angle GCK ſhews by the Table what poligon it agrees to. And two ſemi-baſtions are to be conſtructed at G and K.

4. Let GK be a ſide of the poligon. Biſſect the angle K by the line KC, and biſſect the ſide GK by the perpendicular *m*O, to interſect KC in *n*, and *n* is the center of a regular poligon whoſe ſide is GK; and angle at the center, 2K*nm*. According to which the demi-gorge and flank muſt be ſet off from the Table, and the half baſtion made at K. Again, divide the angle G into two equal parts, by the line GO, to interſect *m*O in O; which muſt be taken for the center of a regular poligon, whoſe ſide is GK, and angle at the center 2GO*m*; according to which the demi-gorge and flank muſt be ſet off at G, from the Table, and the half baſtion A drawn. And this work muſt be performed for all the angles and ſides in the poligon.

5. Otherwiſe. Let AB, AQ, be two ſides of the exterior poligon, biſſect them in *s* and *w*; and draw the two lines *s*O and *wp* perpendicular to theſe ſides AB, AQ, to interſect in *p*. Then *p* will be the center of a regular poligon, whoſe angle at the center is *spw*. Then for a hexagon or more ſides, make $st = \frac{1}{6}$AB, and $wq = \frac{1}{6}$AQ, but take $\frac{1}{8}$ for a ſquare, $\frac{1}{7}$ for a pentagon. Then the demi-gorges and flanks are to be ſet off from the Table; and the lines A*t* and A*q* drawn to form the baſtion at A. And ſo of the reſt.

6. To fortify a ſide that is too long, as MP. This ſide being too long to be defended by muſkets, by reaſon that the baſtions at M and P, are too far

Fig. 44. distant to defend one another; there must be erected in the middle of MP, a *flat bastion* Z, in which the gorge must be about 60 fathoms, and the flanks about 30. And the flanks must be perpendicular to the side MP. And the capital must be equal to the gorge, and the angle of the bastion a right angle. And thus two bastions may be placed on a side that is very long.

7. When a side is too long, but not long enough to require a flat bastion, as PL; it must be fortified with a *curtain retired inwards* as F. In this case, the demi gorges of the bastions at P and L, are $\frac{1}{8}$PL, and the flanks the same, which are perpendicular to the curtain, the retired curtain at F $= \frac{2}{8}$PL, and the flanks of it, $\frac{1}{8}$LP. And these flanks must also be perpendicular to PL.

8. In all sorts of fortification, regular or irregular, for the greater security of the flanks, it is very common to make them hollow as in the bastion at R. This is called a *retired or covered flank*; for it is covered by the butment at the shoulder, called the *orillon*; and this is called a *round orillon* when it is round, and a *square orillon* when it is square. This flank is sometimes an arch of a circle described from the angle of the opposite bastion Q; and sometimes a right line, and then its two ends are equally distant from Q. This retired flank is half the length of the whole flank, by some; and $\frac{2}{3}$ of it, by others, or more. And half of this is allowed for the depth; but some allow but half as much for the depth.

9. When an angle is too acute, it is easily battered down by the enemy's cannon; therefore it must be defended by some additional works. If the adjoining side be long, demi-bastions must be erected on these sides; the greater flank towards the angle, and near double the lesser flank; and both of them perpendicular to the side it stands on.

But

But if the angle be very acute, it may be fortified by cutting off the angle, and making it into a *cut baſtion*, called a *baſtion with a tenaile*; which will be two demi-baſtions, where the longeſt flanks are at the angle; and theſe flanks muſt be perpendicular to the ſides, as you ſee at M. Fig. 44.

10. To fortify an inward angle (of which there is none in this figure); you may conſtruct a *curtain retired inwards*, as is done at F, in the line PL. Or you may make a platform before it, as V in the ſide KL. There are other ways of defending an angle; inſtead of making the platform a parallelogram, it may be made a triangle, &c.

11. Along a river ſide, or by the ſea; it is uſual to have an *indented line* made, which runs in and out; and theſe lines are called *ſaw-works*, as being triangular like the teeth of a ſaw. The lines perpendicular to the ſide are the flanks, being about 20 or 25 fathom, and furniſhed with cannon. Many other methods may be uſed in fortifying irregular ſides and angles, which the experience and judgment of the artiſt will eaſily find out.

As to the different ſituation of places, ſome are advantageous, and ſome the contrary, and want more help. In a town already walled about, the ſeveral parts of the wall will ſerve for curtains; and baſtions are to be made at the angles; and likewiſe flat baſtions, when the angles are too far aſunder. If the wall be bad, and no ditch; it will be neceſſary to circumſcribe the town with a new poligon, and then fortify it according to art.

When an eminence is very near the place, it muſt be taken in with the fortification, or elſe build a caſtle on it, to hinder the enemy from taking poſſeſſion of it. But if it be in the enemy's poſſeſſion, the baſtions muſt be made ſolid, and full, and the parapets raiſed higher than ordinary; ſo that they may command ſuch an eminence, if poſſible.

Fig. 44. When a town is ſituated upon a rock or a mountain, it is hardly acceſſible without great difficulty; and therefore requires leſs fortifying; and it may be ſufficient to fortify the avenues that lead to it, which are generally very few.

If a town be ſituated near a river, that ſide next the river muſt be defended with a rampart, and if it be very long, muſt have one or more flat baſtions placed on it; and ſometimes out-works muſt be made the better to guard the river ſide. Or if the place be ſituated near the ſea, it muſt be fortified by the ſame rules; and to hinder great ſhips from entering the harbour, baſtions muſt be placed on the oppoſite ſides.

Places that are ſituated in an iſland need not be regularly fortified; for the enemy can erect no batteries upon the water. Here the water is a natural fortification, and therefore requires leſs artificial work. And inſtead of baſtions, redoubts that defend one another may be ſufficient.

Places of a high ſituation, have a good air, ſee all the works of the enemy, and command all places round about, and therefore can do more execution. No works of the enemy can be raiſed, but what may be eaſily demoliſhed. The defenders can ſee without being ſeen. Mountains being naturally ſtrong, are fortified more eaſily, and at leſs expence. But on the contrary, it is more difficult to ſhoot downwards with cannon; and there is often a want of water in ſuch places. And theſe places often want room as well as regularity, which makes them very troubleſome to fortify.

Places ſituated in a plain have plenty of field room, where one may fortify to the beſt advantage. It is more eaſy to get neceſſaries in ſuch a place. But then the defenders have no more advantage from the ſituation of the place than the enemy has.

Places

Places ſituated in vallies have all the diſadvan- Fig.
tages poſſible; they are encompaſſed with hills that command the place, whence the enemy will deſtroy all their works, diſmount their cannon, and demoliſh every thing. The fortifying ſuch a place to any purpoſe is an immenſe expence; and therefore it is better to abandon it.

Places ſituated in marſhy grounds are well ſituated, for they can be attacked but in few places, which may be ſoon fortified. The enemy cannot get earth near to make batteries, nor can they make any mines. Yet ſuch places having only ſoft earth, their baſtions often ſink or fall down. And ſometimes the place may be laid under water; or the water that ſurrounds the place, may be let off and drained.

Places near a river or the ſea, have this advantage, that near the water they are eaſily fortified. And the town may eaſily receive neceſſaries by ſhipping. Great ſhips cannot ſtay long to annoy the place for fear of ſtorms; and it requires more force to beſiege and take it. Yet the enemy has the ſame advantage of the ſea as the beſieged, and may prevent ſupplies coming to the town.

In iſlands ſituated at a good diſtance from the ſhore, the people have nothing to fear from the land. And they can eaſily hinder the deſcent of the enemy. Generally theſe places may be fortified with little expence. Yet they may be ſurprized by the enemies fleet, who may do them a great deal of miſchief. But the town has an equal advantage of receiving ſuccours by ſhipping.

The advantages of a good ſituation, are good air and good water. Bad air breeds the plague, and bad water the fever and ſcurvy. The place ought to be ſo high as to command without being commanded. It is neceſſary the earth be good to work with; that its ſituation may make it difficult

Fig. to beſiege; and that it may be ſuccoured in ſpite of the enemy. Near the ſea a good harbour for ſhipping is one of the greateſt advantages; and likewiſe places ſituated upon ſome large navigable river, not far from the ſea, are proper for carrying on trade and commerce.

PROB. III.

To deſcribe the profile or ſection of a fortification, its rampart, parapet, ditch, &c.

The RAMPART.

46. The rampart is a great bank of earth made round a town, to defend and ſecure it from the enemy, as ABCD*b*; the earth it is made of, is taken out of the ditch EFGH. The height of the rampart B*a* is generally about 3 fathom, and its breadth at bottom A*b*, about 15 fathom. The ſides AB and D*b* are ſloped, becauſe earth cannot ſtand if it be built upright; the ſlope within at A is ſuch, that the baſe A*a* is about equal to the height B*a*, and the ſlope without at D is but half as much, or ſuch that $cb = \frac{1}{2}Dc$. But theſe ſlopes depend much upon the conſiſtence of the earth they are built with; for looſe earth requires a greater baſe, or it cannot ſtand. And the ſlope on the inſide may be as great as you pleaſe, that the people may eaſily go up, or convey the guns to the top (called the *terre plain*), for the defence of the place. The rampart is built upon ſome poligon, and at the corners or angles are placed the baſtions alſo made of earth, as deſcribed before. Sometimes theſe baſtions are ſolid earth and ſometimes hollow, to ſerve for magazines to keep ammunition and proviſions.

The PARAPET.

Upon the out-edge of the rampart, or that facing the field, there is raiſed a bank of earth, called

called the *parapet*, or *breaſt-work*, CD*c*. This is about 20 feet thick, and 6 feet high towards the place or on the inſide, and 4 or 5 feet high towards the field. Which difference of height at C and D, makes a ſlope for the muſketeers to fire down into the ditch; *d* is a *foot-bank* to ſtep upon; on the inſide is left a ſpace *b*E called the *berm*, 3 or 4 feet wide, to receive the earth that falls from the parapet, and to hinder it from falling into the moat. The uſe of the parapet is to cover the beſieged, and to defend the men from the enemy's ſhot. All along the parapet at certain diſtances, there are openings called the *embraſures* for the cannon to fire through; they are generally about 12 feet diſtant. Fig. 46.

The DITCH.

The *ditch* or *moat* is a hollow channel made beyond the rampart, as EFGH, and goes round about the place. The level of the ground is EH; this ditch is made at the ſame time the rampart is made, for the rampart is built of the earth that comes out of it. The edges of the ditch EF and HG are made ſloping; and the ſlope EF next the place is called the *ſcarp*; and that next the field HG is the *counterſcarp*. The counterſcarp is made round or circular over againſt the point of a baſtion, the reſt of it is parallel to the reſpective ſides of the baſtions. But the counterſcarp ſometimes means the covert-way and glacis. The breadth and depth of the ditch cannot well be determined, as it depends on ſeveral circumſtances, as particularly upon the nature of the ſoil, whether it is dry or watery. But it muſt be wider than the longeſt trees can reach, and ought never to be leſs than 70 feet.

The edge of the ditch towards the field, over againſt the middle of the curtain, where it is broader than over againſt a baſtion, is terminated by

Fig. 46. by a re-entrant angle, called the *angle of the counterſcarp.* So that each part may be ſeen and defended from the oppoſite baſtion. Ditches may be either dry or filled with water; in dry ditches, there is made a leſſer ditch *f*, about the middle called a *cunette*, which muſt be ſunk to the water. And in ditches full of water, ſtakes or banks of ſand muſt be placed about the middle, to ſtop the enemy's boats.

The COVERT-WAY.

The *covert-way* is a way left upon level ground, beyond the counterſcarp, whoſe breadth is about 4 fathoms, it goes all round by the edge of the moat; it is denoted by HI; it is about 18 feet wide. At the end I is a breaſt-work or parapet 6 feet high, as KI; and a foot-ſtep at I. Upon the covert-way at a little diſtance from the parapet, *palliſadoes* are placed, which are great ſtakes ſet faſt in the ground with ſharp points at the top, and 4 or 5 feet high, and ſo cloſe as one may juſt put a maſket thro'.

The GLACIS.

This is the part beyond the covert-way, as KL, it is made ſloping from the top of the parapet K to the level ground at L, being about 20 fathom. The ſlope KL ought to be ſuch, that the line LK continued will paſs through C.

Beſides the parts mentioned, there are ſome others that are ſometimes made uſe of, as the *falſe bray*, which is a parapet made between the rampart and the ditch, about 6 feet high, and a way left between it and the rampart 3 fathoms wide, called the *chemin des rondes.* From this one may ſee what is done in the ditch.

PROB.

Fig.

PROB. IV.

To deſcribe the out-works of a fortification, and their uſes.

All the works made beyond the ditch are called *out-works*, as *ravelins*, *half-moons*, *horn-works*, *crown-works*, &c. Theſe are deſigned to cover the body of the place, and are made one before another; theſe that are nearer muſt be made higher, ſo as to command thoſe that are further off. The out-works are the moſt important places in a fortification, and are the main ſtrength of it. For however ſtrong a rampart may be, if it is not defended by out-works, it cannot ſtand out long, being continually battered by the enemy. But out-works ſtop the enemy, retard his progreſs, and gain time for the beſieged. I ſhall here lay down the plan of each of theſe; for their heights differ, according to the ſituation of the place.

A RAVELIN.

A *ravelin* is a work placed upon the angle of the 45.
counterſcarp, over againſt the middle of the curtain, as CDE, the angle is towards the field; it is like the fore-part of a baſtion with the flanks cut off. The two faces CD, DE, make an angle of 60 degrees or more. A, B, are two baſtions, CEGF the ditch. Its uſe is to cover the flanks of the baſtions A, B; and alſo a bridge or a gate. There is a ditch on the outſide of it, half the breadth of the great ditch. The faces CD, DE, point to the centers of the baſtions A, B. It has a rampart and a parapet, to be cannon proof; and all theſe are parallel to the faces. They are generally made of earth, but ſometimes for greater ſtrength and laſt, they are walled, eſpecially againſt gates

Fig. gates and bridges, which ought to be well defended. Flanks are ſometimes added to a ravelin, and then they muſt be perpendicular to the curtain; but theſe flanks are of little advantage.

A HALF-MOON.

47. This is an out-work conſiſting of two faces, which make an angle whoſe point is from the field. Its gorges bend in like a bow or creſcent, and is always placed before the baſtion. ABCDE is a half-moon, ED is the counterſcarp, CD and AE are each = half the demi-gorge of the baſtion F, and in the direction of the faces HI, KI; and AB, CB, muſt be parallel to the counterſcarp at E and D, or to the faces of the baſtion HI, KI. On the outſide there is to be a ditch of the ſame bigneſs as in the ravelin. The uſe of this is to cover and defend the point of the baſtion from the enemy's cannon.

A HORN-WORK.

48. *Horns* are placed before the ravelin, towards the field, to cover it and the curtain. Theſe are made ſeveral ways, ſome have long ſides parallel to one another and perpendicular to the curtain; theſe ſides are called *wings*. Some grow narrower, and ſome wider towards the place. But the beſt way is to make them with two demi-baſtions. From the ſhoulders of the baſtions B, D, draw BA, DC, perpendicular to the curtain LM, and equal to 120 fathom. Draw IK for the breadth, which divide into 3 equal parts at F, H. Making AI = IF, and KC = KH; raiſe the perpendiculars FE, HG, each = $\frac{1}{2}$FH, and draw AE, GC, and AEFHGC is the horn-work, which is to cover the ravelin N. There is a ditch on the out-ſide, as in the other works. The horn-work muſt have a rampart and parapet, made of the earth that comes out of the ditch. It will be proper to place a ravelin

lin before the curtain of the horn, to cover its flanks and curtain, conſtructed the ſame way as another ravelin before deſcribed. Fig. 48.

A CROWN-WORK.

A *crown* is a kind of horn-work, made up of two demi-baſtions, and a whole baſtion in the middle. To conſtruct a crown; from the ſhoulders of the baſtion, draw two lines AI, BC, perpendicular to the curtain, and not ſo long as muſket-ſhot: perpendicular to them draw the line AB, ſo that the angle AOI, and BOC may be 30 degrees; or AIO, BCO, 60 degrees; from O (the middle of AB) deſcribe a circle with radius OI, and make the angles IOD, DOC, 60 degrees, or elſe ID, DC = IO. And let I, D, C, be the centers of three baſtions; at D make a whole baſtion, and at I, C, make two ſemi-baſtions, ſuch that the demi-gorge and flank in each be a fourth part of ID, or DC. It muſt have a ditch and rampart, like other works. The uſe of theſe is to cover ſome large ſpot of ground, to ſecure ſome riſing ground from the enemy. The ſides AI, BC, are not drawn to their full length, to ſave room. Theſe are the largeſt ſort of works that are made, and often placed before a ravelin or other works. When the lines AI, BC, are without muſket-ſhot, ſhoulders (or flanks) ſhould be made in the middle of the ſides, which will ſerve to defend the crown-work. 49.

There are other ways of making theſe crown-works, as when the out-lines AI, BC, are not parallel, as when the work is broader at the crown than towards the town, it is called a *ſwallow's-tail.*

There are alſo ſeveral other ſorts of out-works, as

Tenails; ABCDE is a *ſingle tenail*, making a re-entrant angle at C. ABCDEFG is a *double tenail*, making two re entrant angles at C, E. In both theſe 50. 51.

Fig. 50. theſe the ſides AB, DE, or AB, FG, are parallel.

51. A *lunette*, is a work placed on both ſides of a ravelin to ſtrengthen it. It is almoſt in form of a ravelin.

Bonnet, a ſmall work, conſiſting of two faces, making an angle faliant, like a ſmall ravelin.

Prieſts-cap, this is a ſwallow's-tail with two tenails or angles; it conſiſts of 3 ſaliant angles, and 2 re-entrant angles, it is uſed to take in ſprings, and high ground.

Counter-guard, a work ſomething like a ravelin, and placed before a ravelin, or before a baſtion; and has its ſides parallel to that ravelin or baſtion.

Coronment, ſome ſmall additional work to a crown.

Traverſes, trenches with parapets, made ſlanting towards the enemy, and are deſigned to hinder him from paſſing through ſome narrow place or paſſage. And they ſerve to cover the pioneers. Traverſes may be made many different ways, and generally run in ſeveral directions, making many angles.

Redouts, theſe are little ſquare works, with a ditch and parapet, but not ſo ſtrong as to reſiſt cannon. They are uſed to ſecure the lines of approach, and often made on the trenches to ſecure the workmen, or pioneers.

Lines of circumvallation, lines or trenches with a parapet, thrown up by the beſiegers, round the place beſieged, and at a cannon-ſhot diſtance. This is to cover and defend the beſiegers.

Lines of contravallation, are trenches with a parapet, made nearer the town than the lines of circumvallation. Between theſe lines of circumvallation and contravallation, the ſoldiers lie that beſiege the place.

Trenches or approaches, are works carried on by the beſiegers, being cut into the ground, and parapets made of the earth. By theſe the beſiegers gain

gain ground, and draw nearer and nearer the place. And trenches made againſt the beſiegers, to ſtop their progreſs, are called *counter-trenches*. Fig.

Fort, a ſmall fortification, made at a ſmall diſtance from a town; made to ſecure ſome ſpot of ground, or ſome paſs; or to defend the trenches. They are of ſeveral forms, as triangular, ſquare, pentagonal, &c. ABCD is a ſquare fort with half baſtions; and ABCDEF is a ſtar fort with 6 points, made of ſaliant and re-entrant angles. It is a work intrenched on all ſides. 52. 53.

A *citadel or caſtle*, is a ſmall fort, erected in a town, containing 4, 5, or 6 baſtions. It is placed on high ground to command the town, or hinder the approach of an enemy.

Caponiere, a lodgment ſunk 4 or 5 feet into the ground, and raiſed 2 feet above ground. Here 15 or 20 muſketeers are lodged, who fire through *loop-holes* made in the ſides. Theſe are generally made on the glacis or in a dry moat.

Cavaliere, a heap of earth raiſed high, at the top of which is a platform or battery, with a parapet to cover the cannon planted on it.

Battery or platform, a place to plant guns on, it is laid with planks, that the wheels of the carriages may not ſink. They ſlope or deſcend a little towards the parapet, that the guns may not recoil much, and be more eaſily drawn back. The guns are 12 feet diſtant. A *ſunk battery*, is one laid deep in the earth. *Croſs batteries*, thoſe that play athwart. *Murthering battery*, is one that beats upon the back of any place, &c.

Sappe, a deep trench ſunk far into the ground, and deſcending by ſteps, and covered over head. This is uſed when there is a hot fire from the place, to avoid which, the beſiegers work by ſap, that is, under ground till they come to the covert-way and the ditch.

Gallery,

Fig. *Gallery*, a paſſage made from the ſap, acroſs the ditch, and covered over head with trees and earth. This leads to the place where the enemy intends to have a mine.

Mine, a hole dug under ground, being a long paſſage, with many turnings and windings in it; at the end is the *chamber* of the mine, made under ſome place deſigned to be blown up. In this is lodged a quantity of gun-powder for that purpoſe. The chamber is a hollow cube 6 feet every way. If the ground be wet, the powder is in barrels; if dry, in ſacks. They are fired at once by ſeveral *ſaucidges* or pipes of tarred cloth, which reach from the mine to the place where the engineer ſtands to ſpring the mine. The beſieged alſo make ſuch a hole and gallery under ground, to find the enemy's mine, and give it air; and this is called a *counter-mine*.

Enfiled, or *enfiladed*; the ſituation of a poſt that can ſee and ſcour all the length of a ſtreight line or parapet, which by that means is rendered almoſt defenceleſs.

Thus I have deſcribed the principal out-works, and other additional works of a fortification. Theſe out-works are made one before another all manner of ways, to fill up the field. And their greateſt uſe is to advance forward in the campaign, to hinder the ſudden approaches of the enemy, and to ſpend his time before he comes to the main works.

All ditches that are made, ought to communicate with one another, when filled with water; and then one will ſupply another.

SCHOL.

If it be required to find the ſolidity of any of theſe works, and to caſt up the expence requiſite to do them. The content may be found by the rules of Menſuration, and it is no matter whether the

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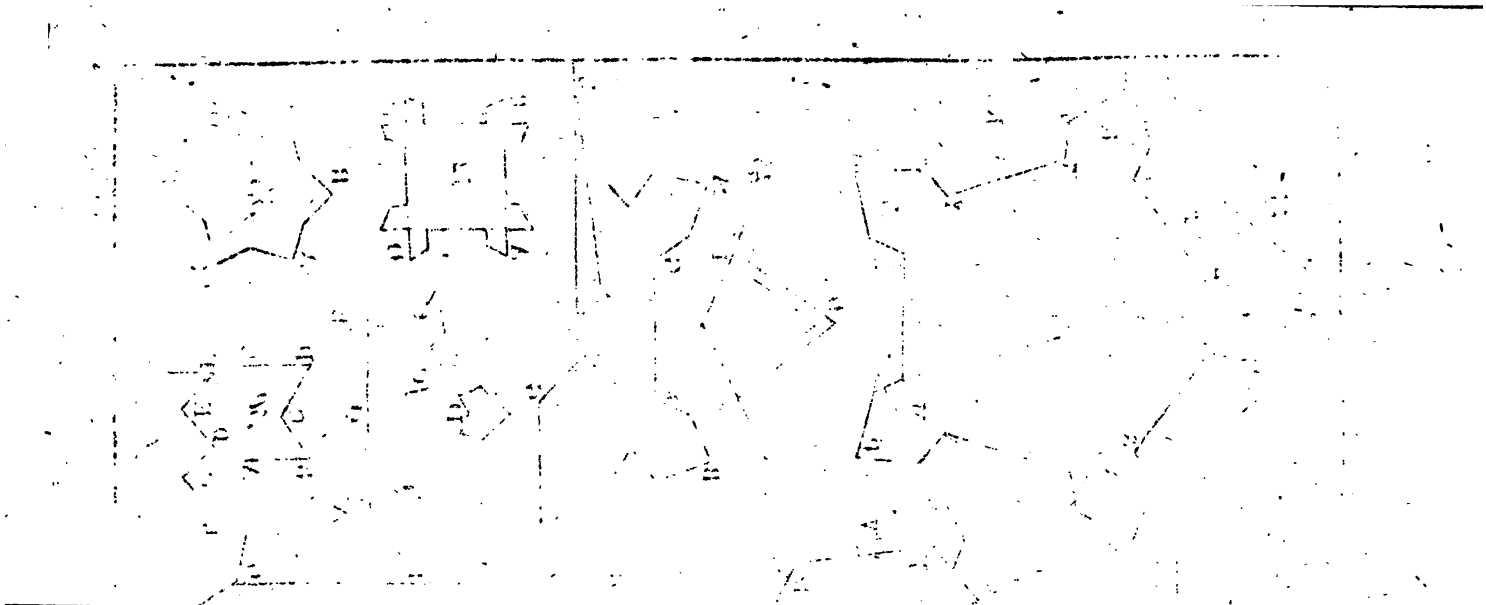


the rampart or ditch be meaſured, for one is nearly equal to the other, as the matter of the rampart once filled the ditch. But ſome engineers ſay, that the earth dug out of a moat, being beat or rammed down, goes into a leſs room by $\frac{1}{12}$ part. Now ſuppoſe the ditch to be EFGH (Fig. 46.) the area or ſection is eaſily found in ſquare yards, and this multiplied by the length of the curtain or rampart, gives the ſolidity of the rampart for that length. But if the breadth of the ditch be not the ſame all the way; take the greateſt ſection and the leaſt ſection, half the ſum gives the mean ſection, for that length of the ditch. Now the ſection multiplied by the length, gives the ſolid yards in that part. And the like for every part, and for every other ditch. Fig.

As to the expence, workmen will dig earth for 2 pence or three pence a yard; but if it is to be carried off, the price will be greater. Likewiſe the time may be computed. For if a man can dig 4 or 5 yards in a day, 1000 men will dig 4 or 5 thouſand yards. And the rule of proportion will tell us how many days 1000 men will be in digging any number of yards.

Laſtly, if you would repreſent any work in perſpective, you muſt follow the rules laid down in Prop. I. *Perſpective*, to which I refer the reader.

PROB. V.

To deſcribe the manner of beſieging, and defending a town.

When proper fortifications have been made about a town, and a ſiege be feared; proviſion muſt be made for a ſufficient quantity of victuals, according to the number of people, and the time of continuance. Alſo the town muſt provide plenty of

Fig. arms and ammunition, as cannon, muſkets, powder and ball, &c. Alſo all ſorts of implements to work with, as ſpades, ſhovels, pick-axes, carriages, &c. and a ſufficient number of ſoldiers to defend the place, as about 300 men for every baſtion, ſo that 2000 may be reckoned a good garriſon, if the place is not very large.

The town being thus ſupplied with all neceſſaries, and the flanks well lined with cannon. All the ſoldiers muſt be called to their particular poſts, the inhabitants ordered to keep in their houſes, obſervations muſt be made on all ſides to diſcover the motions of the enemy.

When an army propoſes to beſiege a town, they are obliged to attack the out-works firſt. In order to this, they muſt pitch upon a proper place, without muſket-ſhot, to begin the line of *contravallation* or *breaking ground*. This line is to be carried about the town, if the garriſon be very ſtrong; or elſe only incloſing a part of it.

When it is probable that an army will relieve the place, a line of circumvallation is to be made further into the campaign, this line is to be carried round about, incloſing ground enough for the beſiegers to lodge in; its parapet is thrown up towards the campaign; and at proper diſtances, are forts and redoubts to be placed, to defend the ſoldiers and workmen.

The next thing is *opening the trenches*, or beginning the *lines of approach*, which are to be carried on obliquely towards the town, ſo as not be enfiladed by any of the outworks.

As ſoon as the beſiegers approach towards the town, the beſieged are to ſhoot off ſeveral cannon, for a ſignal that all may come into the town. And ſeveral ſhot are to be made to keep off light horſemen and ſpies.

The

The beſiegers will endeavour to ſtop all ways, paſſages, and communications of the town and country, to hinder any ſuccours or any intelligence that may be ſent, this is *blocking up* the town. Fig.

While the trenches are drawing near the town, the garriſon ought to make a ſally every opportunity, to annoy them in their works; this will probably draw the army nearer to the town, which may be cannonaded from the town, and perhaps ſome of the principal officers cut off.

When the garriſon is ſtrong, and the enemy attacks any out-works, makes any mines or the like, the garriſon is to *counter-mine* them, and obſtinately defend them, and to make *retrenchments* and *barricadoes* all along, and loſe all things inch by inch.

As the trenches go on, batteries muſt be erected againſt ſallies, and when they come nearer the town, others muſt be made to ruin the defences. At the ſame time the cannon from the town muſt play upon theſe batteries of the enemy to demoliſh them. If the beſieged conſtantly do this, the beſiegers may be forced to raiſe the ſiege.

The beſiegers, while they work in the trenches, muſt be ſupplied with faggots, ſacks of earth, gabions, &c. for defence. And to avoid muſket-ſhot, theſe trenches muſt be carried on in the night, and are beſt carried on by turnings and windings, from right to left, but never to go directly towards any work of the place, for it would be impoſſible to ſtay in ſuch trenches. And near the trenches, bodies of ſoldiers muſt be poſted to defend the pioneers.

When the enemy is come to the glacis, againſt all the efforts of the town. They muſt dig thro' the *counterſcarp*, or work by *ſap*, till they come to the ditch. A battery then muſt be raiſed, to facilitate the paſſing of the ditch; the next thing is to make a gallery, which muſt be defended by the battery. In the mean time the cannon from the town

Fig. will endeavour to deſtroy the enemy's battery; and if it does, it retards the work, which cannot go forward, till the battery be repaired or another erected. If the enemy be ſucceſsful in getting through the ditch, working in the night eſpecially, if they want to blow up the baſtion, the miners muſt be ſet to work to make a mine; and as ſoon as it is ſprung, the beſiegers go and give the aſſault. But if the beſieged by help of their counter-mines, give air to it, it will have no effect.

If making a mine be impracticable, the beſiegers batteries muſt continually play upon the flanks of the fortification, to diſmount their cannon; while other batteries are employed to make a breach, or to break the faces of the baſtion, which being done. The enemy muſt mount the breach, or ſcale the place, and take it by ſtorm. If the beſieged be out of hopes of defending the place, they muſt either ſurrender, or capitulate.

After this manner the whole time of the beſiegers is employed in erecting works of defence, and raiſing batteries, to fire upon and deſtroy the fortifications. While on the contrary, the time of the beſieged is taken up in deſtroying the others works, and driving them back. And this muſt continue till the place is either taken or the ſiege raiſed.

Thus I have given a ſhort account of the manner of proceeding in a ſiege; but my room will not permit to make a longer diſcourſe.

ART.

Fig.

ART. XIII.

GUNNERY.

GUNNERY is an art that teaches how to ſhoot with great guns, by finding what elevation any gun ought to have to hit a mark whoſe ſituation is given; as likewiſe what quantity of powder is requiſite for that purpoſe. It teaches the method of managing all ſorts of artillery, in fighting an enemy, or in defending or attacking a town. It is a proper ſequel to fortification, and both together make but one compleat art, which is the *art of war*. This art of gunnery depends upon the motion of projectiles, which gives rules for finding the velocities, diſtances, elevations, &c. and all requiſites concerned in ſhooting, ſo far as the reſiſtance of the air is ſuppoſed to be nothing, or very ſmall; for when the reſiſtance of the air is conſidered, it makes the Problems of Gunnery too complex to be of any uſe. When the reſiſtance of the air is ſo great that the rules for projectiles do not anſwer, then recourſe muſt be had to tables made from experiments.

DEFINITIONS.

DEF. I.

Random or amplitude, is the diſtance to which a ball is ſhot, from a gun, which is meaſured upon the plane of projection.

DEF. II.

Elevation, is the angle to which the axis of the barrel of the gun is raiſed above the plane of projection.

Fig.

DEF. III.

Altitude, is the height a ball acquires, above the plane of projection, or the higheſt point of its tract.

DEF. IV.

Battery, a place to plant guns on. It is laid with planks and ſleepers, to reſt on; that the wheels of the carriages may not ſink: the guns are generally placed about 12 or 14 feet from one another.

DEF. V.

Charge, is the quantity of powder and ball proper for any gun.

DEF. VI.

Cannon or great guns, are round hollow cylinders, made either of braſs or iron, charged with powder and ball to ſhoot with, and mounted upon carriages, to be eaſily moved from one place to another. There are ſeveral ſizes of them, called by ſeveral names, as whole cannon, baſtard cannon, demi-cannon, or 24 pounders, whole culverin or 12 pounders, demi-culverin 6 pounders, ſakers, minions 3 pounders, drakes and pedereroes, &c. Cannon royal, the bore is 8 inches diameter, 12 feet long; charge 32 pounds of powder, ball $7\frac{1}{2}$ inches diameter, or 48 pounds weight. There are particular terms uſed in Gunnery, as

To tertiate a gun, is to try whether a cannon has a due thickneſs of metal in every part.

To diſpart a cannon, is to ſet a mark on the muzzle-ring to be of an equal height as the baſe-ring; ſo that a line drawn between them, ſhall be parallel to the axis. This mark is for the gunner to take aim by.

Point

Point blank, is the diſtance a ball can be ſhot from a gun, without ſinking below the line of direction; or how far it will go in a direct line. Fig.

The windage of a gun, is the difference between the diameter of the ball and the diameter of the bore.

The parts of a piece of ordnance or cannon, are, the *caliber*, which is the diameter of the bore. *Chamber*, the part where the powder and ſhot lie. *Baſe-ring*, the ring going round the gun at the breech. *Muzzle-ring*, the ring at the muzzle. *Trunnions*, the two knobs or pins that hold the gun in the carriage.

Things belonging to a cannon are, ladles, ſponges, cartridges; alſo needles, thread, twine, nails, hand-ſpikes, crows of iron, baſkets, the pick-ax, mattoc, ſhovel, ſpade, &c. to work with.

Theſe things are to be ſhot out of a cannon. *Bombs*, which are hollow balls of iron; theſe are charged with powder; and a fuzee (which is a compoſition that burns ſlowly) put in at the touch-hole, to laſt all the time the bomb is flying, and after it falls, ſets the powder in fire, which burſts the bomb.

Granadoes, are ſmall bombs or ſhells of iron; or ſometimes made of wood or tin, or even of paſteboard. Theſe are thrown by hand amongſt men.

Red hot bullets, theſe are cannon-balls heated till they have a flame heat, and ſhot out of a cannon, to give light to the field, or to fire a town.

Flaming barrels, or fire barrels, called alſo *thundering barrels*. Theſe are filled with granadoes and fire-pots, or pots filled with gun-powder and a granado; theſe barrels are ſhot out of a mortar-piece, which is a very ſhort gun, with an extraordinary wide bore.

Carcaſſes, are boxes full of powder, made with iron hoops, filled with granadoes and piſtol barrels charged, wrapt up in oiled tow, or other combuſ-

Fig. tible matter, and ſhot out of a mortar. Theſe mortars are elevated pretty high, that the bombs, &c. may fall from a great height.

Pederero or *petterero*, a ſmall cannon to fire ſtones or pieces of old iron out of, when the enemy attempts to board a ſhip, &c.

Petard, an engine to ſhoot with, in form of a high crowned hat; about 8 inches deep, and 5 diameter at the mouth, and 1 inch or 2 at the breech. It is filled with powder, and covered with a plank bound down with ropes, uſed to break down gates, bridges, &c.

Sauſages, are pipes made of tarred cloth, filled with powder, and rolled up in form of a gut, about 2 inches diameter, it is to burn gradually, and reaches from the mine to the engineer, and is ſet on fire when the mine is to be blown up.

Ammunition, ſignifies all ſorts of warlike ſtores, particularly powder and ball.

Artillery, all ſorts of ordnance, or guns great and ſmall, as cannons, mortars, muſkets, &c. all fire arms.

Blunderbuſs, a ſhort gun with a very wide bore, to carry a number of bullets, to ſhoot in a croud.

A *fuzee*, a pipe full of wild-fire put into the touch-hole of a bomb to ſet fire to it.

Shot, all ſorts of bullets for fire arms, from the cannon to the piſtol. Thoſe for cannon are of iron; thoſe for muſkets, carbines, and piſtols, of lead. *Chain-ſhot* and *bar-ſhot*, are two half bullets joined by a chain or iron bar, which cut every thing they meet with.

Chace guns, are thoſe guns in a ſhip that point directly forward or backward.

Swivel guns, guns that move in an iron frame that goes into the ſhip's ſide.

Fire-ball, this is made of powder and other combuſtibles, as big as a granado, wrapt up in tow and

and covered with paper; when primed and ſet on fire, it is caſt into any works in the night to diſcover them. Fig.

To nail up cannon, is to drive a large nail into the touch-hole, or pieces of ſtones; this is to render them uſeleſs to the enemy, when they cannot be carried off. The remedy for this is, to drill a new touch-hole.

PROP. I.

To deſcribe the compoſition, the nature and force of gun-powder.

Gun-powder is but a late invention, ſaid to be found out about the year 1340, or not much ſooner. Before this invention men fought with bows and arrows, and ſuch like things, which did not do a deal of execution. But the dreadful havock which is made by ſhooting balls of iron and lead out of guns of all ſizes, by the force of gun-powder, cauſed all the former ſort to be laid aſide.

Gun-powder is a mixture of charcoal, brimſtone, and nitre; the proportion of theſe is different in different authors. But powder made for the government is compoſed of 10 pounds of brimſtone, 15 pounds of charcoal, and 75 pounds of nitre. Powder made for the uſe of merchants has leſs nitre and more charcoal. At firſt powder was made fine like meal, but afterwards made into corns, which was found to fire quicker than the meal powder. To corn it, it muſt be put through two or three ſieves. The powder is to be moiſtened and reduced to a ſort of paſte, and put into the firſt ſieve, which has large holes, it is broke in pieces by a piece of wood, and paſſes through the ſieve while it is working. The next ſieve which is ſmaller retains the corns, but lets the duſt go thro'.

The

Fig. The corns paſſed through this ſecond ſieve are ſtopt at the third ſieve, with ſtill ſmaller holes. The powder which only goes through one ſieve is uſed by great guns, and called cannon powder. The ſmaller ſort, which paſſes through two ſieves, is uſed by ſmall guns, and called piſtol powder. To glaze the powder, or make it ſhine, they ſhake it in a barrel that has had black lead in it.

The ſudden fulmination of nitre with ſulphur is the principle upon which the action of gun-powder is founded. Sulphur thrown upon melted nitre cauſes an inflammation and an exploſion. The charcoal is only uſed to ſet the mixture ſuddenly on fire.

There is a mixture called *pulvis fulminans*, whoſe exploſion is greater than gun-powder, but it does not fire ſo quickly. It is made thus, 1 part of brimſtone, 2 parts of ſalt of tartar, 3 parts of ſaltpetre; grind them very ſmall, and mix them together well.

The ſudden firing of gun-powder is owing to the nitre, which prodigiouſly increaſes the inflammability of combuſtible ſubſtances. And the vehement exploſion ariſes from the great quantity of air contained in the gun-powder into which it is converted by firing. But if gun-powder be beat ſmall, it will not explode at once, but burn gradually; and therefore in all fire-works, where the powder is to burn gradually, it is beat into duſt.

That gun-powder when fired produces a permanent elaſtic fluid, has been proved by many experiments. And the force of fired gun-powder is the elaſticity or preſſure of the fluid produced by firing it, and is the ſame as the elaſtic force of ſo much air compreſſed into the ſame ſpace with the gun-powder. Likewiſe the elaſtic force of this fiery fluid is proportional to its denſity, as is proved by many experiments. And further the elaſticity

of

of this fluid increaſes by heat, and diminiſhes by cold, like the air; alſo its weight, *ceteris paribus*, is equal to the weight of an equal bulk of air. If a grain of common powder be fired, the fluid produced by it will be 230 or 240 times its bulk, and ſo for a greater quantity. And therefore if this fluid was confined to the ſpace of the gun-powder, it would be 230 or 240 times as ſtrong as common air; that is, its elaſtic force would be 230 or 240 times as ſtrong as the preſſure of the atmoſphere; and that without conſidering any augmentation of that force by heat. But if air be heated to the degree of red hot iron, its elaſticity will be above 4 times as great at that of common air. And therefore the motive elaſtic force of this heated rarified fluid, produced by the firing of gun-powder is 1000 times as great as the preſſure of the atmoſphere. Fig.

It has been diſputed, whether gun-powder fires inſtantaneouſly, or requires time for the flame to run from one part to another. To decide this queſtion, the royal ſociety made ſeveral experiments, the reſult whereof was, that out of a very ſhort barrel 5 or 6 inches long, a quantity of powder was blown out unfired, amounting ſometimes to $\frac{1}{4}$ the quantity of the charge. But if ſhe was charged with a ball, the quantity was leſs, being $\frac{1}{6}$ or $\frac{1}{8}$ of the charge, and this was owing to the time being prolonged by the weight of the ball. But when a gun of the length of 3 or 4 feet was tried, hardly $\frac{1}{20}$ part of the powder was unfired; and when loaded with a ball, the quantity was far leſs, and for the moſt part ſo inconſiderable as to deſerve no notice.

It is evident, being confirmed by all experience, that dry powder fires quicker than damp; and is conſequently ſo much the ſtronger. And powder may be ſo dry, as that the whole may fire before the bullet gets out of the gun; or even before it be

Fig. be ſenſibly moved. Some experiments have been made, which ſhew, that dry powder will ſhoot a ball twice as far as the ſame powder uſed when it is damp.

PROP. II.

To find the velocity of a ball at the mouth of a gun, ſuppoſing all the powder to fire inſtantaneouſly.

54. Let AD be the hollow cylinder of the gun, AB $= a$ the gun-powder, the length AD $= l$, the variable length AC $= x$, $d =$ diameter of the bore, $c = 3.1416$, $b = 16.1$, $w =$ weight of the ball, $p =$ preſſure of the atmoſphere on 1 inch, $f =$ force of the gun-powder at B, $v =$ velocity of the ball at C; then $\frac{cdd}{4}$ = area of the ſection, and $\frac{cddp}{4}$ = preſſure of the atmoſphere on the bullet. Then $f = \frac{1000cddp}{4}$, by the laſt Prop. And $x : a :: f : \frac{af}{x}$ = force on the bullet at C. Then (by Cor. Prob. I. Sect. III. Fluxions,) $v\dot{v}$ is as $\frac{F\dot{s}}{b}$. And in the caſe of falling bodies, $v = 2b$, $\dot{v} = \dot{x}$, and $F = w$, $\dot{s} = \dot{x}$; therefore $2b\dot{x} : \frac{w\dot{x}}{b} :: v\dot{v} : \frac{af}{x} \times \frac{\dot{x}}{b}$, or $2b : w :: v\dot{v} : \frac{af\dot{x}}{x}$; therefore $v\dot{v} = \frac{2bafx}{wx}$, and the fluent is $\frac{v^2}{2} = \frac{2baf}{w} \times b.$ log. x. But in B, $x = a$; therefore $\frac{vv}{2} = \frac{2baf}{w} \times b.$ log. $\frac{x}{a}$.

$\frac{x}{a}$, and when x comes to be equal to AD, $\frac{vv}{2} =$ Fig. 54. $\frac{2baf}{w} \times h.\ \log.\ \frac{l}{a}$; therefore $vv = \frac{4baf}{w} \times h.\ \log.\ \frac{l}{a}$, and $v = \sqrt{\frac{4baf}{w} \times h.\ l.\ \frac{l}{a}} = \sqrt{\frac{4ba}{w} \times \frac{1000cddp}{4} \times h.\ \log.\ \frac{l}{a}} = \sqrt{\frac{1000cbaddp}{w}} \times \text{hyp. log. } \frac{l}{a}$.

Cor. *In all guns, the ſquare of the velocity is as* $\frac{add}{w} \times \log.\ \frac{l}{a}$, or $\frac{a}{d} \times \log.\ \frac{l}{a}$.

EXAM. 1.

Let the length of the gun be 3 feet, diameter of a leaden ball 1 inch, a the charge of powder 2 inches, then $w = .214$, $p = 14.7$, and hyp. log. $\frac{l}{a}$ or $18 = 2.890$, then

log. 1000	3.00000
log. $c = 3.1416$	0.49714
log. $b = 16.1$	1.20682
log. $add = .1666$ —	1.22182
log. $p = 14.7$	1.16732
log. 2.890	0.46089
	5.55399
ſub. log. $w = .214 =$	1.33041
	6.22358
half —	3.11179
velocity —	1296 feet in a ſecond.

EXAM. 2.

Let the length of a gun be 9 feet, the diameter of the bore, or rather of the iron ſhot 4 inches, its

Fig. 54. its weight w 9 pounds, $a = 8$ inches the charge of powder, hyp. log. $\frac{l}{a}$ or $13.5 = 2.602$,

log. 1000	3.00000
log. c, 3.1416	0.49714
log. b, 16.1	1.20682
log. p, 14.7	1.16732
log. *add*, 10.66	1.02775
log. 2.602	0.41530
	7.31433
log. w, 9	0.95424
	6.36009
half —	3.18004
velocity —	1514 feet in a second.

PROP. III.

To find the velocity of a ball at the mouth of a cannon or any gun; supposing the powder to fire gradually, so as to be all fired when the ball is at the mouth of the gun.

We must first find the relation between the time and space described. The force at any instant will be as the quantity of powder fired, divided by the space it is contained in; therefore putting the quantities as in the last Prop. we shall have the force as $\frac{t}{x}$. Now suppose $x \propto t^n$, to find n. By Prob. I. Sect. III. Fluxions, $\dot{v}v \propto F\dot{x}$, and $F \propto \frac{t}{x} \propto \frac{x^{\frac{1}{n}}}{x} \propto \frac{1}{x^{\frac{n-1}{n}}}$, therefore $\dot{v}v \propto \frac{\dot{x}}{x^{\frac{n-1}{n}}}$, and the

the fluent $v^{\frac{1}{n}} \propto nx^{\frac{1}{n}}$. Also $\dot{x} \propto \dot{v}t \propto v \times \frac{1}{n}$ Fig. 54. $x^{\frac{1}{n}-1}\dot{x} \propto x^{\frac{1}{2n}} \times x^{\frac{1}{n}-1}\dot{x} \propto x^{\frac{3-2n}{2n}}\dot{x}$, and the fluent $x \propto x^{\frac{3}{2n}}$, or $x^1 \propto x^{\frac{3}{2n}}$, whence equating the indices, $1 = \frac{3}{2n}$, and $2n = 3$, and $n = \frac{3}{2}$. Therefore $x \propto t^{\frac{3}{2}}$. This being known, put $g =$ whole time of passing through AD.

We shall now have $f = \frac{1000cddp}{4}$ as in the last Prop. for the force at the first firing, and $\frac{af}{l} =$ force at D when it is all fired. And to find the force at C; since it is generally as $\frac{t}{x}$, we shall have $\frac{g}{l}\left(\frac{\text{time}}{\text{space}}\right) : \frac{af}{l}$ (force at D) :: $\frac{t}{x}\left(\frac{\text{time}}{\text{space}}\right)$: force at C $= \frac{aft}{gx}$. Then since $v\dot{v} \propto F\dot{s}$, the body being given; and in a falling body $v = 2b$, $\dot{v} = \dot{x}$, $F = w$, $\dot{s} = \dot{x}$; therefore $2b\dot{x} : w\dot{x} :: \dot{v}v : \frac{aft\dot{x}}{gx}$. Therefore $v\dot{v} = \frac{2baft\dot{x}}{gwx}$. But $l : g^{\frac{3}{2}} :: x : t^{\frac{3}{2}} = \frac{g^{\frac{3}{2}}x}{l}$, and $t = \frac{gx^{\frac{2}{3}}}{l^{\frac{2}{3}}}$; therefore $v\dot{v} = \frac{2baf}{gwx} \times \frac{gx^{\frac{2}{3}}\dot{x}}{l^{\frac{2}{3}}} = \frac{2bafx^{-\frac{1}{3}}\dot{x}}{wl^{\frac{2}{3}}}$. And the fluent is $\frac{vv}{2} = \frac{2bafx^{\frac{2}{3}}}{\frac{2}{3}wl^{\frac{2}{3}}}$, or $vv = \frac{6bafx^{\frac{2}{3}}}{wl^{\frac{2}{3}}}$, and $v = \sqrt{\frac{6bafx^{\frac{2}{3}}}{wl^{\frac{2}{3}}}}$. And when $x = l$, then $v = \sqrt{\frac{6baf}{w}} =$

$\sqrt{6ba}$

Fig. 54. $\sqrt{\frac{6ba}{w} \times \frac{1000cddp}{4}} = V = \sqrt{\frac{\frac{3}{2} \times 1000\,chaddp}{w}}$.

Several Corollaries follow from this ſolution, as

Cor. 1. *The laſt velocity at* D *will be the ſame whatever length the gun be of; provided the height of the charge be given, and the powder be of ſuch a goodneſs as to be all fired, juſt when the ball reaches* D.

For l does not come into the computation, and a remaining the ſame, V will remain the ſame.

Cor. 2. *The powder being the ſame, the final velocity will be as the ſquare root of the height of the charge.*

For the reſt remaining, the velocity will be as $\sqrt{a}$.

Cor. 3. *If l be the length of gun, wherein the charge of powder will all be fired at the muzzle* D. *Then in any ſhorter gun* AC, *the ſame powder and charge remaining, the ultimate velocity will be as* $\sqrt[3]{\frac{x}{l}}$.

For the reſt remaining, V is as $\sqrt{\frac{x^{\frac{2}{3}}}{l^{\frac{2}{3}}}}$.

Cor. 4. *And in this laſt caſe, the velocity with the charge a will be no greater than with the charge* $\frac{x^{\frac{2}{3}}}{l^{\frac{2}{3}}}\,a$ *or* $a\sqrt[3]{\frac{xx}{ll}}$.

For let y = the charge that will juſt be fired at C. Then by ſuppoſition $a : y :: g : t :: l^{\frac{2}{3}} : x^{\frac{2}{3}}$, whence $y = \frac{ax^{\frac{2}{3}}}{l^{\frac{2}{3}}}$; and the ſuperfluous powder does nothing.

Cor. 5. *Suppoſing as in Cor.* 3. *and in any gun x longer than l, the velocity at laſt will alſo be as* $\sqrt[3]{\frac{x}{l}}$.

Cor.

Fig. 54.

Cor. 6. *In guns of different bores, the reſt remaining as before, the velocity at the muzzle will be as* $\sqrt{\frac{a}{d}}$.

For it is as $\sqrt{\frac{dda}{w}}$ or $\sqrt{\frac{dda}{d^3}}$ or $\sqrt{\frac{a}{d}}$.

EXAM. 1.

Let the gun be 3 feet long, $d = 1$ inch, the diameter of the ball of lead, $a = 2$ inches of powder, then $w = .214$, $p = 14.7$.

log. 1000	3.00000
log. c, 3,1416	0.49714
log. b, 16.1	1.20682
log. $add = .1666$ —	1.22182
log. 1.5	0.17609
log. p 14.7	1.16732
	5.26919
log. w .214 : : 1 —	1.33041
	5.93878
half	2.96939
velocity at D =	932 feet *per* ſecond.

EXAM. 2.

If the length of a gun be 9 feet, diameter of her ſhot being iron = 4 inches, weight $w = 9$ lb. $a = 8$ inches the charge.

Fig. 54.

log. 1000	3.00000
log. c, 3.1416	0.49714
log. b, 16.1	1.20682
log. p, 14.7	1.16732
log. add, 10.66	1.02775
log. 1.5	0.17609
	7.07512
log. w, 9	0.95424
	6.12088
half	3.06044
velocity =	1149 feet *per* ſecond.

PROP. IV.

Suppoſing the powder in a gun to fire inſtantaneouſly; to find the depth of the charge, to produce the greateſt velocity poſſible.

It is ſhewn in Prop. II. that the velocity in ſuch a ſuppoſition is $\sqrt{\frac{1000cbaddp}{w}} \times b.\log.\frac{l}{a}$. And ſuppoſing all things given except the charge a, then a log. $\frac{l}{a}$ will be a maximum, put x for a the variable quantity, then x log. $\frac{l}{x}$ = maximum, put $y = b$ log. $\frac{l}{x}$, then xy = max. and $x\dot{y} + y\dot{x} = 0$. The fluxion of $\frac{l}{x}$ is $\frac{-l\dot{x}}{xx}$, and this divided by $\frac{l}{x}$; gives $\frac{-l\dot{x}}{xx} \times \frac{x}{l} = \frac{-\dot{x}}{x}$, therefore $\dot{y} = \frac{-\dot{x}}{x}$; and $x\dot{y} + y\dot{x} = x \times \frac{-\dot{x}}{x} + y\dot{x} = 0$, or $-\dot{x} + y\dot{x}$

$+ y\dot{x} = 0$, and $y = 1 = \dot{b}.\ \log.\ \frac{l}{x}$; that is, Fig. 54. 2.30258 × com. log. $\frac{l}{x} = 1$, and com. log. $\frac{l}{x}$ $= .4343$; and $\frac{l}{x}$ = num. of log. $.4343 = 2.718$; therefore $2.718\ x = l$, and $x = \frac{l}{2.718}$, or $x = .3679\ l$, when the velocity generated is a maximum.

SCHOL.

This can be of no uſe in any gun, except they be exceeding ſhort; for ſuch a charge will burſt any gun, of a moderate length, to pieces.

PROP. V.

To find the due quantity of powder proper for the charge of any gun.

This muſt be found by experiments for ſome particular gun. Take any gun, and a ball proper for it, and firſt let it be charged with a quantity of powder equal to $\frac{1}{3}$ the weight of the ball, and elevating the piece to any given angle, ſuppoſe 45 degrees, let it be diſcharged, and let the random or diſtance of the ſhot be meaſured. Again, let it be charged with powder equal to half the weight of the ſhot, and diſcharged, and the random meaſured. And laſtly, let the charge of powder be $\frac{2}{3}$ the weight of the ſhot, and the random alſo meaſured. Then the ſeveral diſtances or randoms compared, will ſhew which is the moſt convenient charge. For though the greateſt charge of powder ſhoots the furtheſt, yet when the increaſe of diſtance is but ſmall; it is better to take up with the leſſer charge, becauſe ſo much powder is ſaved.

Fig. But there are ſo many irregularities attends this practice, that it is proper to repeat the experiments over and over; and laſtly, to take a mean for each charge. For in the ſame piece, with the ſame charge, and placed at the ſame angle of elevation, there is ſuch difference in the diſtances, that one cannot but be ſurprized at it. And therefore it is neceſſary that ſeveral experiments be made with the ſame charge. By comparing many experiments together, I take that to be the moſt proper charge, when the weight of the powder is half the weight of the ball.

The proper charge for one gun being known, the charge for any other gun is known by this rule.

Take the charges of powder in proportion to the cubes of the diameters of the balls, or as the cubes of the diameters of the bores of the guns.

EXAM.

If a ball 4 inches diameter requires $4\frac{1}{2}$ *lb.* of power; how much powder will a ball 6 inches diameter require?

4
4
16
4
64 : $4\frac{1}{2}$ :: 6
6
36
6
216
$4\frac{1}{2}$
864
108
64)972(15 2 *lb.* anſwer.
64
332
320
12

Some

Some people ſay, that guns ought to be loaded according to their weight; and give it as a rule, that a quarter of a pound of powder ſhould be allowed for every hundred weight of the gun, if it is of iron; but ſomething more for a braſs gun. Fig.

But the quantity of powder ought to be ſo proportioned to the ball, as the buſineſs to be done requires, or according to the execution to be done. And modern engineers, that have had a deal of experience, tell us, that the charge of powder ought not to exceed $\frac{1}{3}$ the weight of the ball. For although greater velocities are generated by greater charges, yet by the great reſiſtance of the air, theſe greater velocities are ſoon reduced to the velocities generated by the leſſer charges.

But guns may be charged by meaſure as well as by weight. For a pound of powder fills 31.06 cubical inches; therefore ſay, as 1*l.* to 31.06 inches; ſo is *half*, or a third, of the weight of the ball, to the charge in cubical inches.

It muſt be obſerved, that there is great difference in the goodneſs of powder; and even the ſame powder will be ſtronger or weaker, according as it is kept dry or moiſt; for powder will imbibe moiſture, and that abates the ſtrength of it. And if powder be known to be weaker than ordinary, more of it muſt be taken for a charge.

Note, powder for proof, (that is, for trying the ſtrength and goodneſs of guns) muſt be double the quantity of powder for ſervice, as ſome ſay; and others ſay, as 3 to 2.

When a gun is to be charged with a *cartridge* (which is a bag, or roll of canvaſs, paper or parchment, of the bigneſs of the gun's bore, to hold juſt a charge); having the weight of a charge of powder in pounds, multiply it by 31.06, and call the product p. Let d = diameter of the bore in

Fig. inches. Then $\frac{p}{.7854dd}$ = length of the cartridge.

Some authors tell us, the charge of powder for great guns must be 2 diameters of the ball; and for lesser guns 3 diameters.

PROP. VI.

To describe the construction and use of a great gun.

A gun is said to be well fortified, when there is sufficient thickness of the metal at the breech, and in every other place. For if a gun be badly fortified, or too thin of metal, it will endanger bursting, and in such, the charge ought to be moderate. When a gun is well fortified, the diameter of the bore must be to the diameter of the gun at the breech, as 2 to 7. And then the diameter of the bore being 2, the thickness of the metal will be $\frac{5}{2}$ or $2\frac{1}{2}$. And therefore the diameter of the bore will be to the thickness of the metal as 1 to $1\frac{1}{4}$, or as 4 to 5.

Hence the circumference of the gun at the touch-hole, is 11 diameters of the bore. For if the diameter of the bore be 2, the diameter of the gun then is 7, and the circumference 22, and 2 is to 22 as 1 to 11. The circumference at the touch-hole being 11, the circumference at the trunnions must be 9, and at the muzzle 7. And this is esteemed the true fortification for an iron gun.

But for brass guns, the circumference at the breech must be 9 diameters of the bore, 7 at the trunnions, and 5 at the muzzle. Therefore the thickness of the metal at the breech, is about $1\frac{1}{11}$ the diameter of the bore.

And the length of either sort is to be 7 diameters of the gun at the breech, though this is not so

ſo material for the length, which upon occaſion may be different. Fig.

But the rule here laid down for a gun truly fortified ſeems only to be proper for one ſort. For the thickneſs of the metal is eſtimated the ſtrength of the gun, and ought to be proportional to the force within; but that force is as the quantity of powder, and that is as the ſquare of the diameter of the bore. Therefore in wider pieces, the metal ought to be thicker than the rule makes it. For if the bore of an iron gun be 6 inches, and $7\frac{1}{2}$ inches (by the rule above) be the proper thickneſs of the metal. Then for a bore of 8 inches, 10 inches would be the thickneſs of the metal, which is too little. For it ſhould be as $36 : 64 :: 7\frac{1}{2} : 13\frac{1}{3}$ inches for the true thickneſs of the metal, for 8 inches bore.

But if the gun be thinner on one ſide than the other, her ſtrength muſt be eſtimated by the thineſt ſide. Or if there be flaws or places eaten away with ruſt, called *honey-combs*, theſe weaken a gun very much, and if they be deep, ſuch guns ought not to be uſed.

To adjuſt the ball to the piece, you muſt uſe ſuch a ball, that the difference of the diameters of the ball and bore may be about $\frac{1}{20}$ the diameter of the ball; and this ball will ſuit the gun. If the difference be leſs than that, the ball (being of iron) is in danger of ſticking, and the gun burſting. And if it be more, a deal of the ſtrength of the powder will be loſt, in paſſing by the ball. This allowance is called the *windage*.

How great guns are to be managed in ſhooting. Firſt the gun being placed upon her carriage, ſhe muſt be charged with powder; to do which the gunner makes uſe of a *ladle*, which is a long ſtaff with a tin box at the end, which being filled once,

Fig. or oftener, with powder taken out of the *budge barrel*, is a juſt charge for the gun.

Next the wad is to be put in, and rammed down with the *rammer*, which is a long pole on purpoſe.

Then the ſhot muſt be taken and tumbled into the gun, going cloſe to the wad.

Then by help of the quadrant and a ſtreight rule laid to the inſide of the bore, the piece muſt be elevated to a proper angle.

Then by help of a *priming iron*, which is a long ſmall piece of iron, a hole is made through the cartridge to the powder, and the powder is to be poured into the touch-hole to the top, and covered with a thin plate of lead, called the *apron*.

And when ſhe is to be fired, a hot iron is applied to the touch-hole, which ſets the powder on fire, by which ſhe is diſcharged.

Then the gun is to be ſcoured with a ſponge, that no fire remain in her. A *ſponge* is a long ſtaff with a roll at one end, covered with a ſheep ſkin, of the bigneſs of the gun's bore. Great guns muſt be cooled leiſurely with water after every ſhot; for being much heated by often ſhooting, they endanger burſting.

The platform of a battery ought to be firm; for if it ſhake when a gun is fired, the ſhot will be very uncertain.

In every battery, there muſt be half as many ſpare guns, as there are mounted; to ſupply thoſe that are diſabled.

It is a very important improvement in gunnery, called by the French, ſhooting by *ricochet*. Which is, to ſhoot with ſmall charges of powder, and elevating the gun, ſo that the ſhot may juſt go over the parapet, and drop into their works. By this means more miſchief is done than in firing directly againſt the works.

Some

Some authors have given Tables of Gunnery, Fig. shewing the name, weight, length, diameter of the bore, weight of shot and powder, &c. of which the following is one, for all sorts of ordnance.

name.	weight gun.	length gun.	diam. bore	weight shot.	weight powder.	point blank.
	Cwt.	feet.	inch.	poun.	poun.	feet.
cannon 8	70	12	8	60	30	900
cannon 7	$60\frac{1}{2}$	11	7	42	21	900
demi-cann.	$50\frac{1}{4}$	10	$6\frac{1}{2}$	32	16	900
culverin	40	12	$5\frac{1}{3}$	18	10	900
demi-culv.	30	10	$4\frac{1}{2}$	11	8	890
saker	20	9	$3\frac{3}{4}$	6	4	800
minion	12	7	3	$3\frac{1}{4}$	$2\frac{1}{2}$	600
falcon	6	7	$2\frac{3}{4}$	$2\frac{1}{2}$	$2\frac{1}{4}$	600
falconet	$3\frac{1}{2}$	6	$2\frac{1}{4}$	$1\frac{1}{3}$	$1\frac{1}{4}$	520
rabonet	$2\frac{1}{2}$	$5\frac{1}{2}$	$1\frac{1}{2}$	$0\frac{1}{2}$	$0\frac{3}{4}$	500
base	$1\frac{3}{4}$	$4\frac{1}{2}$	$1\frac{1}{4}$	$0\frac{1}{2}$	$0\frac{1}{2}$	400

PROP. VII.

To find how much powder will fill any bomb or granado shell.

RULE.

Cube the inner diameter of the shell, and divide by 59.3, the quotient will be the pounds of powder that will fill the shell.

For the shell is in form of a globe or sphere, let d = the diameter; then $\frac{11d^3}{21}$ = solidity in inches, and therefore $\frac{11d^3}{21 \times 31.06}$, that is, $\frac{d^3}{59.3}$ shews the pounds of powder.

In a shell the lowest part is the thickest and heaviest, and therefore will fly foremost, and it will fall

Fig. fall upon that ſide, and not on the fuzee, which would put it out.

EXAM.

Let the diameter of the ſhell be 13 inches,

13	59.3)21.97(37.05 *lb.*	
13	: : 17 79	the anſwer.
39	4180	
13	4151	
169	29	
13		
507		
169		
2197		

PROP. VIII.

Having given the diameter of an iron or leaden ball; to find the weight of it; and the contrary.

RULE.

As 100 to the cube of the diameter in inches; ſo is 14 to the weight in pounds, for the iron ball. And the leaden ball will weigh half as much more.

EXAM. I.

Suppoſe the diameter of the ſhot to be 6 inches,

6	100 : 14 : : 216	
6	14	
36	864	
6	216	
216	100)3024(30.24 *lb.* for iron.	
	15.12	
	45.36 for lead.	

EXAM.

Fig.

EXAM. 2.

If an iron ball weigh 48*lb.* what is the diameter?

14 : 48 : : 100 :

14)4800(342.8, whose cube root is 7,

42

60

56

40

28

120

so the diameter is 7 inches.

Note, the weight of a ball of stone is $\frac{2}{5}$ of the weight of an iron ball.

PROP. IX.

To find the distance to any fort or battery, by the firing of a gun there.

Sound is known to move 1140 feet in a second of time, but the motion of light is instantaneous. Therefore observe with a pendulum how many seconds there are between the flash of fire and the report of the gun; this number of seconds multiplied by 1140, gives the number of feet for the distance from the place where the gun is fired.

Or *thus*, count how many double beats of a common watch passes between the firing of the gun, and the sound reaching your ear; multiply this by 490, gives the distance in feet.

Or thus, if your pulse be regular, count the number of pulses between the flash and the report; which multiply by 970, gives the distance in feet.

EXAM.

Fig.

EXAM. 1.

I obſerved a gun fired at a certain place, and by obſervation found, that ſeven double beats (or 14 ſingle ones) of my watch paſſed between the firing and the ſound, what is the diſtance?

490
7
3430 feet the diſtance.

EXAM. 2.

Another perſon counted the number of pulſes to be $3\frac{1}{2}$.

979
$3\frac{1}{2}$
2910
485
3395 feet for the diſtance.

PROP. X.

Beſides the reſiſtance of the air, there is another irregular force by which a projectile is made to deflect from a direct courſe. To explain this deflection, and cure it.

It is hardly poſſible for a ball to be ſhot out of a gun, without its rubbing againſt one ſide or another of the barrel; and the friction it receives by that means, gives it a whirling motion round an axis, which is always perpendicular to the axis of the barrel or to the tract of the ball. The conſequence of this is, that one ſide of the ball meets with a greater reſiſtance of the air than the other ſide; and the air acting obliquely againſt that ſide with the greater reſiſtance, will force it to move towards

wards that ſide where is the leaſt reſiſtance; and Fig.
the quantity of this deviation will be as the difference of the reſiſtance of one ſide above the other. And conſequently the ball will always deflect towards that ſide of the barrel where the friction happened; for that ſide of the ball being retarded in its motion, meets with the leaſt reſiſtance. But it is impoſſible to know before-hand, on which ſide of the barrel the friction will happen. But when the ſhot is over one may nearly determine on which ſide it was. For if the ſhot be over the mark, it is on the upper ſide; if ſhort, on the under one. If on the right or left, it is on the right or left ſide reſpectively. And beſides, this tract of deviation muſt be a curve line. For as this diſturbing force is continual; every ſucceeding part of the tract will deviate (the ſame way) from the former part; which is the nature of a curve line.

I have been long acquainted with this irregular force and its effects, which I found by experience when I uſed to practice ſhooting; and it preſently occurred to me, that the greater reſiſtance on one ſide of the ball, was the true cauſe of its going out of the line of direction. And to ſatisfy myſelf about it, I ſuſpended a wooden ball loaded with lead, in a ſtring, and tied it to a tree that hung over a river, that it might play freely in the ſtream, and noting the place where it reſted, I then twiſted the ſtring, by turning the globe often about, and putting it into its former place, it reſted but a little while; for as the ſtring began to untwiſt, it moved gradually towards the ſide which conſpired with the motion of the water. And being at its furtheſt extent, it reſted till the motion began to diminiſh, and then it came gradually back to its firſt place, and reſted there till the motion of the globe twiſted the ſtring the contrary way; and then it moved to the other ſide. And thus it made

ſeveral

Fig. ſeveral vibrations to and fro, till the motion was ſpent, and then it reſted in the firſt place. This I ſeveral times repeated.

I alſo tried the ſame in a ſtrong wind, with the like ſucceſs; for the ball always deviated from the plane of the winds motion towards that hand where it was leaſt reſiſted.

Now to remedy this reflection, one way is, to uſe bullets that are not round, but oblong, ſomething like a ſlug. But then they ought to be turned in a lath or throw; that the fore end may be regular, and all ſides alike; that the air may act equally on all ſides. Such a body as this, ſhot out of a gun, cannot by friction be made to revolve about an axis, and therefore that irregular force and its effect will be prevented.

Another way to prevent this deflection is to make the guns riffe bored; theſe rifled barrels are made with ſeveral threads of a ſcrew running ſpiral ways on the inſide of the barrel; between theſe threads are channels cut in the bore, all which muſt be exactly parallel to one another, and make about one revolution in the length of the barrel, going uniformly about. The number of theſe threads are different, according to the wideneſs of the barrel.

There are different methods of charging theſe pieces, one is this. After the powder is put in, they take a bullet ſomething bigger than the bore of the gun, and greaſe it well, and putting it into the mouth of the piece, they ram it down with an iron rammer, hollow at the end; in ramming down the bullet the ſpiral threads enter and cut into the bullet, and cauſe it to turn round in going down, and being ſhot out, it follows the ſame direction of the rifles; which cauſes it to turn round an axis parallel to the gun's bore.

Another way is to charge them at the breech, where there is a hole to put in the powder and ball, and

and then a ſcrew ſcrews in to fill up the hole. But Fig.
ſome barrels ſcrew off at the breech, to be charged. These guns are made ſtronger at the breech than common; and it is plain they can only be uſed for lead bullets, for iron will receive no impreſſion.

And thus a bullet ſhot out of a rifled barrel, beſides its direct motion, gains a motion round the axis of the gun, by which the reſiſtance on the foreſide of the bullet will be the ſame on all ſides; for if it ſhould be greater on one part than another; that part, by the circular motion, is preſently transferred to the oppoſite ſide, and then it acts the contrary way; and ſuch irregularities rectify one another; ſo that the ball will always go right forwards.

This may be explained by the motion of an arrow; for if an arrow that is not feathered, be ſhot from a bow, its motion will be very irregular; for if it be the leaſt crooked imaginable, it will move towards that hand where the concave ſide lies. But when it is feathered truly, to give it a circular motion and make it ſpin, the concave part is turned every way, ſo that it will always fly ſtreight forward. See Exam. 36th Fig. 220, of my Mechanics, 4to.

But in your common guns that are not rifled, I know no way to prevent that deflexion, but to poliſh the inſide of the barrel, and oil the bullet when it is charged; for by this means the friction within the barrel will be made as ſmall as poſſible; except you chuſe to ſhoot with an oblong bullet as before-mentioned.

PROP. XI.

Let d = diameter of an iron ball, $q = 7$, its denſity, $p = .00117$, the denſity of air, b = the velocity of the ball at firſt, v = velocity after it has deſcribed

Fig. *ſcribed the ſpace* x. $T = \frac{8dq}{3bp} = \frac{16000d}{b}$. *Then if* x *be any ſpace which the bullet deſcribes after it is fired,* $\frac{v}{b}$ *is the number of the logarithm* $\frac{-x}{36841d}$; *all in feet.*

This is demonſtrated in Cor. 2. Prob. XIII. Sect. III. Fluxions. Suppoſing the reſiſtance of the air to be as the ſquare of the velocity, and ſuppoſing the elevation to be ſmall, that the tract may not differ much from a right line.

Suppoſe the diameter d of the ball to be 6 inches; then $d = \frac{1}{2}$, and $\frac{v}{b}$ is the number of the log. $\frac{-x}{18420}$. And ſuppoſe the ſpace deſcribed x to be 200, 400, 600, 800, 1000, &c. feet, and b the firſt velocity to be 1. To find the velocity at the end of theſe ſeveral diſtances. The quantities reduced, we ſhall have $\frac{-x}{18420} = -1 + \frac{18420 - x}{18420}$ for the logarithm, and the number of it $\frac{v}{b}$ or v.

x.	Logarithm.	Number v.
feet 200	— 1.98914	.9753
400	— 1.97829	.9512
600	— 1.96743	.9277
800	— 1.95657	.9048
1000	— 1.94571	.8825
1200	— 1.93485	.8607
1400	— 1.92400	.8395
1600	— 1.91314	.8187
1800	— 1.90228	.7985
2000	— 1.89143	.7788
2200	— 1.88056	.7596
2400	— 1.86970	.7408

&c.

By

By this computation it appears, that if the ſpaces described be taken in arithmetical progreſſion, increaſing; the logarithms of the velocities will be in arithmetical progreſſion decreaſing. For the differences of the numbers in the firſt col. x, are all equal; and ſo are the differences of the logarithms in the 2d column. And of conſequence the numbers (v) in the third column, are in geometrical progreſſion. And this is agreeable to what Sir J. Newton has laid down in Prop. V. Book II. Principia. Fig.

Hence let the firſt velocity be what it will, an iron ball 6 inches diameter, after having moved 800 feet in the air, will proceed with $\frac{9}{10}$ its firſt velocity, having ſcarce loſt $\frac{1}{10}$ of that velocity. And after moving through 2400 feet, will have loſt little more than a quarter of its firſt velocity.

When $x = 18420$, then the log. is 1.00000, and the number of it $v = \frac{1}{10}$. So after it has moved 18420 feet, it ſtill has $\frac{1}{10}$ its firſt velocity.

If x be greater than 18420, as ſuppoſe it be 100000, then the logarithm is $= -1 - \frac{81580}{18420}$ $= -1.44289$; that is -2.55711, and the number $v = .03607$. So that except x be infinite, the velocity cannot be quite taken away.

Again,

Suppoſe the diameter of the ball 8 inches $= \frac{2}{3}$; then will $\frac{-x}{36841d} = \frac{-x}{24561} = -1 + \frac{24561 - x}{24561}$ for the logarithm of the velocity. If x be 1067, then the logarithm is -1.95657, and the velocity $v = .9048$.

Again,

If the diameter of the ball be 2 inches $= \frac{1}{6}$, then $\frac{-x}{36841d} = \frac{-x}{6140} = -1 + \frac{4140 - x}{6140}$, for

Fig. the logarithm; put $x = 267$; then the logarithm is -1.95656; and the number of it, or $v = .9048$ for the velocity.

By all which it appears, that the ſame velocity remains in each ball, after deſcribing ſpaces proportional to their diameters; as it ought to be by this law of reſiſtance.

For the time.

The time may alſo be had from the ſame Corol. For $v = \frac{Tb}{T+t}$, and $Tv + tv = Tb$, and $tv = Tb - Tv$, whence $t = \frac{b-v}{v} T$.

EXAM.

Let the diameter of the ball be 6 inches as before; then $T = \frac{16000d}{b} = 8000$. And let the velocity be .8607, then $b - v = 1 - v = .1393$, and $t = \frac{.1393}{.8607} \times 8000 = 1294'' = 21\frac{1}{2}$ minutes for the time of flight, through 1200 feet. But this is upon ſuppoſition, that the firſt velocity is 1. But if the firſt velocity be $1000 = b$, then $T = \frac{8000}{1000} = 8$, and then $t = \frac{.1393}{.8607} \times 8 = 1.''294 = 1\frac{3}{10}''$, the time of deſcribing 1200 feet.

In this Prop. I have ſhewn the motion of an iron ball through the air, when its tract is not much different from a right line; and how much motion it has loſt in moving through any given ſpace, by the air's reſiſtance, and likewiſe the time of deſcribing that ſpace; all which is founded upon the theory of reſiſtance laid down by *Sir I. Newton*. Therefore the reſiſtance and retardation of a projectile cannot be much greater than is here determined;

mined; from whence it will follow, that conſider- Fig.
ing the ſmall retardation made in the motion of an iron ball, the path deſcribed by a projectile cannot differ very far from a parabola, eſpecially in ſlow motions.

In Prob. XIX. Sect. III. *Fluxions*, I have calculated the tract or curve in which a projectile moves in a reſiſting medium; by this Prob. its place may be found in that tract, its velocity, and time of moving, if any body will be at the pains to go through the computation. But as it can be of no manner of uſe in gunnery, by reaſon of its difficulty, I ſhall ſay no more about it.

The line deſcribed by a ball in the air, is a curve of the hyperbolic kind, one of the aſymptotes is parallel to the line of projection, and the other is perpendicular to the horizon. But the calculation of ſuch a curve is difficult, which makes it alſo uſeleſs in Gunnery. And although by the reſiſtance of the air, the curve deſcribed by a ball may differ conſiderably from a parabola; yet ſince the computations by it are ſo ſimple, and ſo eaſily done, we ought not to ſet the rules aſide as uſeleſs, but rather try to correct them from experiments, eſpecially as we have no other fixt rules to go by. For this end the following Prop. will be of uſe.

PROP. XII.

To conſtruct tables for ſolving the common Problems of Gunnery.

To do this, we muſt have a competent number of experiments, made with all ſorts of guns at all elevations, to get their randoms, in manner following. Take any gun, of which you know the proper charge, and elevate her to 5 degrees, and fire her off, and meaſure the random, which ſet

Fig. down. Then elevate her to 10, 15, 20, 25, &c. degrees, as far as 90, and find the random for each elevation as before; all which muſt be ſet down in a Table. But one ſhot for each cannot be relied on, but ſeveral muſt be made, in order to take a mean among them; ſo that the Table may be as perfect as poſſible.

This being made for one ſort of guns, you muſt do the ſame for guns of all ſizes, and make a Table for each, and then you will have a ſet of Tables, as compleat as the nature of the thing will admit of. It is true this requires a great deal of labour and time.

When any Problem in Gunnery is propoſed, and reſolved by the rules derived from the parabola, and the ſolution compared with a ſimilar one, which may be had from theſe Tables; the error of theſe rules will appear, and afford means to thoſe that are poſſeſſed of ſuch Tables, of correcting the parabolic theory, which is the only one we have to calculate by. Therefore ſince we have no methods of computation, but what are founded on the parabola, and ſince they are ſo extremely eaſy in themſelves, and may be corrected ſo as to come near the truth. I ſhall here therefore ſolve the common Problems of Gunnery by the parabolic hypotheſis, and ſhew how to proceed with ſuch Tables for a ſolution, with which I ſhall conclude this Article.

PROB. I.

The random or horizontal projection, at any elevation being given; to find the random at any other elevation, the charge remaining the ſame.

RULE.

Fig.

RULE.

The horizontal randoms, made with the same velocity, at different elevations; are as the sines of the doubled angles of elevation.

EXAM.

Suppose the random at 20 degrees elevation, be 3058 feet, to find the random at 62 degrees.

As S. 40°	—	9.80806
to 3058	—	3.48543
S. S. 124 (or 56)		9.91857
		13.40400
to 3944 (anſ.)		3.59594

To ſolve it by the Tables of random. Seek among the Tables, till you find one that has the ſame elevation and random, or the neareſt thereto; then in that Table againſt the other elevation, you have the random ſought. This random you may compare with the other found by calculation, the difference will be the error.

Cor. 1. *The greateſt random is at the elevation of* 45°.

For the ſine of 90 (2×45) is the greateſt ſine.

Cor. 2. *The greateſt random is equal to the latus rectum of the parabola.* 55.

For let L = latus rectum, then ſince the angle BAE = 45°, AB = BE = 2BD, but $L \times DB = AB^2$, that is $L \times \frac{1}{4}AB = AB^2$, or $L = 2AB = AC$.

Or thus, ſince $DB = \frac{1}{4}AC$, B is the focus, and AC the latus rectum.

SCHOL.

It is obſerved, that the greateſt random is not quite 45°, in the air, but only about $44\frac{1}{2}$.

Fig.

PROB. II.

The greatest random being given; to find the random at any given elevation.

EXAM.

Suppose the greatest random 5080 feet, to find the random at 25 degrees elevation.

As radius	—	10.
to 5080	—	3.70586
S. 50	— —	9.88425
Anſ. 3892	—	3.59011

PROB. III.

Having the random and elevation of the piece given; to find the elevation proper for hitting a mark, at a given diſtance.

EXAM. 1.

Suppose the random 4900 feet, and elevation 42°; and the diſtance of the mark 3520 feet.

As 4900	—	3.69019
S. 84	— —	9.99761
ſo 3520	—	3.51654
		13.54415
Anſ. S. 45 36, or 134 25		9.85396

Half is 22 48, or 67 12, which are the two elevations, at either of which the mark will be hit.

EXAM. 2.

Suppose the greatest random 5080 feet, and the mark 3520 feet diſtant.

The

The greateſt random is at 45°, therefore Fig.

as 5080	—	3.70856
to rad.	—	10.
ſo 3520	—	3.54654
S. 43 52 or 136 8,		9.84068

Half of which is 21 56, or 68 4; ſo that the mark may be hit, either at the elevation of 21 56, or the elevation 64 4.

By the Tables; find the given diſtance and elevation in ſome Table, in which, againſt the given diſtance, is the elevation.

Cor. *Hence the randoms are equal, when the elevations are equally diſtant above and below* 45°, *being the complements to one another.*

SCHOL.

It has been obſerved, that in the air, the lower randoms go a little farther than the correſpondent upper randoms. Alſo when the greateſt random is given from experiments, the randoms at leſſer elevations, being computed by the ſines of the doubled arches, always come out leſs, than the randoms obtained from experiments.

PROB. IV.

Having the greateſt random upon a horizontal plane; to find the greateſt random on an inclined plane.

RULE.

The projections or randoms made on different planes, with the ſame velocity, are in the complicate ratio of the ſines of the angles which the lines of direction make with the plane and zenith; and the ſquares of the coſines of the plane's elevations reciprocally.

Fig. And the greateſt randoms on different planes, are as the verſed ſines of the angles between the plane and zenith directly, and the verſed ſines of double theſe angles reciprocally; either for aſcents or deſcents.

Or rather thus; the randoms are reciprocally as the verſed ſines of the ſupplements of the angles between the plane and zenith.

EXAM.

Suppoſe the greateſt horizontal random 3990, what is the greateſt random on a plane elevated or depreſſed 9 degrees.

For the aſcent.

Here $\frac{v.90}{v.180} = \frac{1}{2}$, and therefore $\frac{1}{2} : \frac{v.81}{v.162} :: 3990$: random ſought $= \frac{2 \times v.81^{\circ} \times 1990}{v.162}$, for the aſcent.

Log. 2.	—	0.30103
verſ. 81 (.84357)	—	1.92611
3990	—	3.60097
		3.82811
verſ. 162 (1.95105)		0.29027
Anſ. 3450	—	3.53784

For the deſcent.

The random ſought $= \frac{2 \times v.99^{\circ} \times 3990}{v.198 \text{ or } 162}$

Log. 2.	—	0.30103
ver. 99 (1.15643)		0.06311
3990	—	3.60097
		3.96511
verſ. 198 (1.951)		0.29027
Anſ. 4730	—	3.67484

Or

Fig.

Or thus.

Rad.	——	10.
3990	— —	3.60097
		13.60097
verſ. 99°	—	10.06312
345°	— —	3.53785
		13.60097
verſ. 81	—	9.92612
4730	— —	3.67485

Cor. *The greateſt random on an inclined plain, is when the line of direction biſſects the angle between the zenith and the plain. And the randoms are equal which are made at equal diſtances above and below this line.*

PROB. V.

Having the greateſt horizontal random, to find the elevation to hit an object with the leaſt force, whoſe height and diſtance are given.

The angle of elevation is found thus; ſay as the diſtance, to radius; ſo the altitude of the object, to the tangent of its elevation. But this elevation may be more eaſily had by obſervation. Then ſuppoſe a line drawn from the place to the object; then the line of direction ſought muſt biſſect the angle between this line or the object, and the zenith. And this will be ſo whether the object be above or below the plain of the horizon.

Cor. *To hit an object with the leaſt force, the line of direction muſt biſſect the angle between the zenith and the object.*

PROB.

Fig.

PROB. VI.

The velocity in feet, and the elevation being given; to find the horizontal random.

RULE.

Let $f = 16.1$ feet, A = fine of double the angle of elevation, v = the velocity. Then $\frac{vvA}{2f \times \text{rad.}}$ = horizontal random.

EXAM.

Let the elevation be 30 degrees, and the velocity 350.

350	—	2.54406
350	—	2.54406
S. 60	—	9.93753
		15.02565
32.2	—	1.50785
rad.	—	10.
		11.50785
Anſ. 3295	—	3.51780

Cor. 1. *The elevations being the ſame, the horizontal randoms are as the ſquares of the velocities.*

§6. Cor. 2. *The velocity is a mean proportional between* 16.1 *and the parameter to* AF, *at the point of projection.*

For if AG be the velocity, or ſpace moved in a ſecond; then GH = 16.1, the ſpace deſcended in a ſecond. But $\frac{AG^2}{GH}$ = parameter to AF = $\frac{vv}{f}$.

Cor.

Cor. 3. *The horizontal velocity is conſtantly* AI. *And the principal latus rectum of the parabola* $= \frac{AI^2}{f}$. Fig. 56.

Cor. 4. *The velocity at any point* A *of the ſame parabola, is as the ſecant of the angle* IAG, *or reciprocally as the coſine of the elevation.*

For AI being conſtant, AI : AG : : rad. : ſecant of IAG : : coſ. IAG : to radius.

Cor. 5. *If* DE *be made equal to* $\frac{1}{4}$ *the latus rectum, and the horizontal line* GEH *drawn; the velocity at any point* F *in the curve, is that which a body acquires by falling from the line* GF *to that point* F. *Thus the velocity at* D, *is that by falling through* ED, *and at* F *through* GF, *&c.* 57.

For if DE $=$ DO $=$ $\frac{1}{4}$ the latus rectum, then O is the focus, and OK or DI $=$ 2DE $=$ 2IK. The velocity in falling through ED is the ſame as in falling through DO; but in an equal time it will deſcribe 2DO or DI, or OK, and therefore will paſs through the point K of the parabola. And therefore the body will deſcribe the parabola whoſe focus is O, and $\frac{1}{4}$ the latus rectum ED; and conſequently the velocity at F is that which is acquired by falling through GF. But (by the Conic Sections) GF is $\frac{1}{4}$ the latus rectum to the diameter FN.

Cor. 6. *A body projected perpendicularly upwards, with the velocity at* F *in the parabola* DFA, *will aſcend to the height* FG *or* PE.

PROB. VII.

Having the random made with a certain charge of powder, at a given elevation; to find the charge of powder, for any other random, with the ſame elevation.

RULE.

Fig. 57.

RULE.

The charges of powder will be nearly as the randoms, excepting that the charge is something too great, when it is less than the given charge; and too little, when it is bigger. See Cor. Prop. II. But if Prop. III. be admitted, the random will be exactly as the charge.

EXAM.

If 8*lb.* of powder throw a ball 4010 feet; how much powder will throw it 5700 feet?

4010*f* : 8*lb.* :: 5700*f*

8

4010)45600(11.3*lb.* answer.

4010

5500 (but something

4010 too little by Prop. II.)

1490

PROB. VIII.

The height of the perpendicular projection and elevation, being given; to find the height of the shot.

RULE.

The altitudes of projections are as the versed sines of the doubled angles of elevation, when the velocities are given; and as the squares of the velocities, when the elevation is given.

EXAM.

Let the height of the perpendicular projection be 3420 feet, what is the height when the elevation is 35 degrees?

As

As verſ. 180	—	10.30103	Fig.
3420 —	—	3.53402	57.
verſ. 70	—	9.81821	
		13.35223	
alt. 1125	—	3.05120	

Cor. 1. *If* P = *perpendicular projection in feet, then* $v = 2\sqrt{Pf}$ *the velocity.*

For $\sqrt{f}$ (height) : $2f$ (velocity) : : $\sqrt{P}$: v.

Cor. 2. *If* B = *verſ. of twice the elevation,* $\frac{vvB}{8f} = \frac{PB}{2\,rad.}$ = *the altitude.*

Cor. 3. *The perpendicular projection is the greateſt altitude.*

For the verſ. of 180 = 2 radius.

PROB. IX.

Given the height of the perpendicular projection, and the elevation; to find the random.

RULE.

As radius, to the perpendicular projection; ſo the ſine of twice the elevation, to half the random.

For the random = $\frac{2AP}{r}$.

EXAM.

Let the perpendicular projection be 3420, and the elevation 35°.

Rad.	—	10.
3420	—	3.53402
S. 70	—	9.97298
½ random, 3214,		3.50700
random 6428		

Cor.

Fig. 57. Cor. *The perpendicular projection is half the greatest random.*

SCHOL.

It has been obſerved, that from the reſiſtance of the air, the perpendicular projection is ſomething more than half the greateſt random.

PROB. X.

The random and elevation given; to find the height of the ſhot.

RULE.

As radius, to the tangent of elevation : : $\frac{1}{4}$ the random, to the height of the projection.

56. For AB : BE or 2BD : : rad. : tan. BAE : : $\frac{1}{4}$AB : BD.

EXAM.

Let the random be 6428, and elevation 35.

Rad.	——		10.
tan. 35	—	—	9.84522
$\frac{1}{4}$ tan. 1607	—	—	3.20601
height, 1125,	—	—	3.05123

PROB. XI.

The random and height of the ſhot being given; to find the perpendicular projection.

RULE.

Let D = random, and $d = \frac{1}{4}$D, b = height of the projection; then $b + \frac{dd}{b}$ = height of the perpendicular projection.

This appears by Cor. 6. Prob. VI.

EXAM.

Fig.

EXAM.

Suppose the random be 6428, and height 1125.

Then $d =$ 1607

1607

11249
9642
1607

1125)2582449(2295
2250 1125

3324 3420 anſ.
2250

1074
1011

63

Cor. 1. *The principal latus rectum of the parabola is* $\frac{DD}{4b}$.

Cor. 2. *The velocity of projection is* $2\sqrt{fb}$.

PROB. XII.

Given the elevation, and random; to find the time of flight.

RULE.

The times of flight are as the ſines of the angles of elevation.

Or thus, as rad. : tan. elevation : : random : Q.

Then $\sqrt{\frac{Q}{f}} =$ time in ſeconds.

EXAM.

Fig.

EXAM.

Let the elevation be 35, and random 6428.

Rad.	— —	10.
tan. 35	—	9.84522
6428	— —	3.80807
Q	— —	3.65329
$f = 16.1$	—	1.20682
		2.44647
Anſ. 16.7^{ll}	—	1.22323

Cor. *The height of the ſhot is* $\frac{1}{4}$Q, *being the height of the parabola.*

SCHOL.

58. The rules here laid down are moſtly derived from the motion of projectiles in my *Mechanics*. But beſides the arithmetical calculations, the ſeveral Problems may be ſolved geometrically, after this manner. Let AB be the perpendicular projection. About AB deſcribe the ſemicircle ACB. Make AK = 2AB, which will be the greateſt random. Let AH be any other random, and make AE = $\frac{1}{4}$AH, and erect EC*c* perpendicular to AH to cut the ſemicircle in C, *c*; and draw ACD, A*cd*. Make AF = $\frac{1}{2}$AH, and raiſe FD*d* perpendicular to AH. Biſſect FD, F*d* in G and *g*, which will be the vertexes of two parabolas paſſing through A, H, and touching the lines AD, A*d*, and whoſe parameters are 4BL, 4B*l*. Therefore if EAC, EA*c* be two elevations, ſo that CR = *c*R, the projections made in theſe two directions will deſcribe theſe parabolas, and both paſs through H. And if AI = $\frac{1}{4}$AK, and IR touch the circle in R, AR will be 90°, and the projection made along AR will paſs through the point K. So then AGH, A*g*H, will be the paths of the projectiles, in direction AC

or

or Ac, and whoſe velocity is that acquired by falling through BA. And the angle ACB or AcB is always a right angle. And the heights of the parabolas will be $FG = EC$ or AL, and $Fg = Ec$ or Al, being the verſed ſines of twice the elevation. And the time of aſcent and deſcent in AB, to the time in the parabola AGH (or AgH), is as AB to AC (or Ac). Alſo LC (lc) is the ſine of twice the elevation $= \frac{1}{4}$ the random. Fig. 58.

When any place is to be bombarded, it may be done either by elevating the piece to a certain angle, and increaſing or diminiſhing the quantity of powder, till the ball be juſt thrown to the mark. Or elſe by taking a proper charge of powder, and elevating or depreſſing the piece till ſhe juſt ſhoots to the place; and then keep to that charge and elevation. If the gun is elevated to 45 degrees, the leaſt charge of powder will then do the buſineſs; and by this, much powder will be ſaved. In bombarding houſes, it is beſt to elevate the gun above 45°, for then the bomb riſes to a greater height, and ſo will do more miſchief. But where the bomb falls in the ſtreets, or is ſhot among a parcel of men, it is better to elevate below 45°, for then it will not bury itſelf in the ground. And at ſea, mortars ſhould be elevated above 45°, for the agitation of the ſea continually changes the direction of the mortar, and would otherwiſe render the ſhot very uncertain; but in great elevations, ſeveral degrees alteration make very little difference in the effect.

Fig.

ART. XIV.

ARCHITECTURE;

OR THE

ART OF BUILDING.

THIS art teaches how to erect any kind of buildings, as palaces, churches, or private houſes. In performing of which, many hands are employed of various branches of buſineſs, as maſons, ſtone-cutters, bricklayers, carpenters, joiners, ſmiths, plumbers, glaziers, tylers, ſlaters, &c. This is an art of ſuch extenſive uſe and benefit, that there is no ſuch thing as mankind living without it; the inhabitants of all countries have contrived methods to build themſelves houſes to live in; by which they are preſerved from cold, ſnow, froſt, rain, and all inclemencies of weather; and in which they can preſerve from deſtruction, all their victuals, cloaths, bedding, and all other neceſſaries they make uſe of. This art then muſt be ranked in the firſt place, for till that has been duly put in execution to procure a dwelling for any perſon, he can practiſe no other art.

In every building three things muſt be principally conſidered, *ſtrength*, *conveniency*, and *beauty*. *Strength* requires good materials and good work. *Convenience* requires, that the ſeveral parts be ſo ordered and diſpoſed that one part may not be a nuiſance to another; that thoſe parts which one has the moſt recourſe to, ſhould be the neareſt at hand; that the room or bigneſs of every part be according to the uſe of it, or the buſineſs to be done in it.

Beauty

Beauty ariſes from the form of it, and the correſpondence of all the parts with the whole, ſo as to be agreeable to nature. Fig.

When a building is to be erected, the expence of it ought to be carefully calculated, by ſome judicious perſon, and timely proviſion made for money, that the workmen may not ſtay for their wages beyond the due time. And a ſufficient ſtock of materials muſt be provided and got in, that there be no ſtop in going forward with the building, for it is an advantage to the work to be carried on and finiſhed with expedition. The materials to be taken care for are timber, ſtone, bricks, lime, ſand, and iron. As to the choice or goodneſs of theſe, and the manner of uſing them, I ſhall explain in the following Propoſitions.

PROP. I.

To deſcribe the ſeveral materials, as timber, ſtone, lime, ſand, &c.

The ſorts of timber moſtly uſed in building are oak, elm, aſh, and fir. The oak will warp and caſt, except it be many years old, and thoroughly dry; but the elm will caſt more than any other; the aſh not ſo much. But fir is the beſt wood for ſtanding its lift, for it is of a ſpringy nature, and will caſt very little. All ſappy parts muſt be cut out of wood uſed in building, for it ſoon decays. The moſt proper time of felling timber is the winter ſeaſon, and ſome ſay in the decreaſe of the moon. In the winter the ſap is all expended in feeding the branches, and wood felled then will neither warp nor cleave; but felled in ſummer it will rift and ſhrink. It is therefore a wrong method of felling oak in the ſummer, for the ſake of the bark; for the wood will not be ſo ſtrong nor ſo

Fig. good. A better way is to bark the trees in the ſummer ſtanding, and let them ſtand till the winter. For after the bark is taken off, the trees will live and thrive till winter. The timber of ſuch trees as grow in open fields, is better than ſuch as grow in cloſe woods and ſhady places. Some ſay, that the faſter the tree grows, the ſtronger the timber will be; and that good ground produces better timber than bad ground. The age of a tree may be known, when it is ſawn through, by the number of rings round the pith; in each ring there is a number of open pipes, which convey the ſap upwards. As many of theſe rings, ſo many years old the tree is. If you obſerve whereabouts the rings are wideſt; that ſide of the tree faces the ſouth. No timber ſhould be uſed in building till it be three years old. Fir wood is the beſt for roofs, and joiſts, and for flooring and doors. But oak is the beſt for window-frames or door-poſts, or any part that faces the weather, for it will laſt beyond any other ſort of wood.

Stone is of various kinds, as marble, alabaſter, fire ſtone, purbeck ſtone, free ſtone, common quarry ſtones, pebbles, or river ſtones, Portland ſtone, &c. There are ſeveral ſorts of *marble*, as white, black, grey, green, veined or ſpotted. It is a very hard ſtone, and capable of a fine poliſh; it is uſed for chimney-pieces, hearths, tables, tomb-ſtones, and ſtatues. *Alabaſter* is a ſoft, white ſort of marble; it is uſed for monuments and carved figures, or to make cement. *Fire-ſtone*, called alſo rigate ſtone, is uſed for hearths, ovens, ſtoves, &c. *Purbeck ſtone* is a hard grey ſtone, uſed for pavements, they may be cut ſquare and poliſhed. *Free ſtone*, this is dug out of quarries, being a ſoft whitiſh ſtone, and of a fine ſandy greet, and may be ſquared and wrought into any form. It is uſed for flags, jaumbs for windows and doors, and in

building

building the fronts of houſes, and the arches of bridges, &c. Fig.

Quarry ſtones, are dug out of a rock or quarry; they will not work to any form, being very hard. They are very good for building common walls. *Pebbles* are ſuch ſtones as lie looſe in the bottom of a river, of all forms, and very hard. They are uſed for common pavements in the ſtreets, and for coarſe walls. *Portland ſtone*, is brought from Dorſetſhire, it is ſofter and whiter than Purbeck, and raiſed in greater blocks out of the quarry. This is much uſed in building, but it will not endure fire.

Lime, this is made of burnt ſtones, and theſe ſtones are either dug out of quarries, or gathered out of rivers. The quarries proper for this are called lime-ſtone quarries. That lime is beſt which is made of the hardeſt ſtone, and when burnt is of a blueiſh colour. Quarry ſtones are better than looſe gathered ſtones. Lime may be made of ſhells, as cockles, oyſters, &c. And about London they frequently burn chalk for lime; but it makes very weak lime; and is fitteſt for plaiſtering, being very white. When the lime ſtones are well burnt, which requires two days or more, it is taken out of the kiln, and led away. Then it is to be *ſlaked*, which is done by turning it over with a ſhovel, and throwing as much water upon it, little by little, as will make it fall to powder. When it is all thus ſlaked, it muſt be lightly covered with ſand. It ought to lie thus for a week at leaſt, before it is made up; but if it lie longer it is better; for it requires time for all the knots to diſſolve. When it is to be made into mortar, it muſt be mixed with ſand; if it be river ſand, you muſt put two load of it to one load of lime, but pit-ſand being ſofter, take three load of it to a load of lime. The lime and ſand muſt be well mixed together,

Fig. and made up with water. But it is not to be uſed directly, but muſt lie ſome time.

Sand, this is had ſeveral ways; by digging in the ground, or leading it out of the rivers, or from the ſea. The river ſand makes the ſtrongeſt lime, but leſs of it is uſed. Sea ſand is reckoned the worſt, by reaſon the ſalt makes it moulder away, therefore not ſo fit to make lime with. Sand is bad when dirt is mixed with it.

Metals, which are made uſe of in building, are iron, lead, and copper. *Iron* ſerves to make nails, crooks, bars, hinges, grates, &c. Engliſh iron is the worſt for making nails of, &c. for it is ſo brittle the nails will not clinch. Iron is good, if it appears of a fine grey colour when broken; but bad, when many glittering ſpecks appear in it. *Lead* ſerves for glazing with, and ſheet-lead ſerves for covering the roofs of churches, and any ſtately buildings, and for lead gutters, and pipes for carrying off the water. And is ſerviceable for faſtening any iron work in ſtone. *Copper* is uſed for making nails, claſps, &c. Alſo *braſs* is uſed much in the ſame way.

Bricks are a ſort of artificial ſtone. They are made of clay, dug and ſeaſoned in the winter, and burnt in the ſummer. There are ſeveral ſorts of bricks, as *common bricks*, from 9 to 11 inches long. *Hollow bricks*, flat on one ſide and hollow on the other, when the hollows are laid together, they make conduits for conveying water away. *Flemiſh bricks*, a ſmall ſort for paving with. *Feather-edged bricks*, are thinner on one ſide than the other. *Paving bricks*, or *paving tiles*, of different form and ſizes, for laying floors with. *Place bricks* and *ſtock bricks*, for walling. *Statute bricks*, ſuch as are made about *London*, according to act of parliament, they are to be 9 inches long. *Coping bricks*, for the tops of walls. And ſeveral other ſorts.

PROP.

Fig.

PROP. II.

To lay the foundation of any building.

The greateſt care ought to be taken to lay a good foundation, for if that fails, the ſuper-ſtructure cannot be ſupported. And generally ſpeaking, there are more errors committed this way, than in any thing elſe, and are of the moſt pernicious conſequence, being often attended with the ruin of the whole building. Foundations are to be laid within the ground, and ſo much the deeper as the earth is ſofter and looſer; and you muſt dig till you come at firm ground. The beſt foundation is when we build upon a rock, the next is a hard and firm gravel; a ſtrong ſtiff clay is alſo very good; but ſoft looſe earth is bad; and the worſt is mooriſh ſoft earth; and among quick-ſprings. Sound ground may be known by ſtriking forcibly with a heavy body, and then if it be firm it will not ſhake or ſound, or move the water in a veſſel. The foundation muſt be digged wider than the thickneſs of the wall, to make offsets, and ſo much wider as the ground is looſer; they muſt be very level at the bottom; and in laying your ſtones, place the flateſt ſide downwards upon the level earth, for if you ſet an angle or edge of it downwards, the weight will make it ſink into the earth like a wedge, and make the building ſhrink. You muſt make offsets at the bottom; ſometimes two or three muſt be made. But never lay a ſtone farther out for an offset than a third part of its length. The looſer the ground is, the more offsets muſt be made; for obſerve, the greater the baſe on which your foundation ſtands, and the more points the foundation ſtones touch in, the firmer it is, and the more weight it will carry. The offsets muſt all

Fig. 59. be lower than the floor within. ABCDE is a wall with its offsets. Some workmen, after the foundation is laid, begin directly to build upon it, without filling up the hollows or vacancies at the bottom; but this is wrong; for when the outside ſtones are laid, the inſide muſt alſo be filled with ſtones truly laid; and when it will hold no more, earth muſt be rammed in between them, and alſo under ſuch ſtones as will not lie level, and the earth well beat. If this is not done, it cannot be called a good foundation.

In ſoft boggy ground which ſinks much, piles muſt be driven into all the foundations, as thick as they can ſtand, and then planks laid over them and pinned faſt, upon theſe you begin to build; and this is abſolutely neceſſary in laying the foundations of the piers of bridges, where the ground is not ſound. And the piles muſt be driven to the ſound ground.

In a large houſe the foundations muſt be twice as thick as the walls; and as much offset made on one ſide as on the other, or rather ſomething more on the outſide.

If the ground is bad only in ſome particular places, and the reſt good; arches may be turned over the bad places, which will take off the weight,

PROP. III.

To deſcribe the method of walling.

The materials for walling with are ſtone and brick, and mortar made of lime; and in walling with either, obſerve theſe particular directions. 1. The walls of any houſe muſt be exactly perpendicular. 2. The ſtrongeſt and greateſt materials muſt be at the bottom. 3. The walls of a houſe muſt diminiſh proportionally in thickneſs, every

ſtory

ſtory in height. 4. That for every foot and half, Fig.
or two feet high, a courſe of *throughs* be laid; 59.
theſe throughs go quite through the wall, and ſerve to bind it together, which otherwiſe would ſplit. 5. The corners muſt be well tied together, with *quoins* laid croſs and croſs, of a good length, theſe quoins are commonly ſquared ſtone. 6. No wall ſhould be wrought above three feet high before the next adjoining walls be brought up, and theſe in their turn muſt be carried above the reſt, but not more than three feet. If the walls are not thus bound into one another, they will crack and part. 7. In walling, obſerve to break the joints, that is, not to make one joint above another, which is bad workmanſhip, and may cauſe the walls to crack; for the ſtones or bricks ought all to be tied over one another as much as they can. 9. Avoid making great joints, but lay the bricks or ſtones as cloſe as they can be put, with narrow joints of lime. 10. In all buildings, the ſolid parts muſt be directly over the ſolid parts, and the vacant parts over the vacant. Thus pillars muſt be above pillars, windows and doors above windows and doors, and walls above walls, partitions above partitions; to make good work.

Brick walls are built, by laying the bricks lengthways upon the wall, end for end, in a due quantity of lime; this is called a *ſtretching courſe*. And when the wall is raiſed about every ſix courſes, then a courſe of bricks is laid with their lengths croſs the wall, called a *heading courſe*. This binds the wall together. But when a brick does not reach through the wall, a ſtretching courſe more is laid on one ſide of the wall than on the other. And then two bricks laid croſs muſt tie over one another, for all the length of the wall. Sometimes the wall is ſo thick, that 3 or 4 bricks muſt be thus tied together croſs. Stone walls are built much

Fig. 59. much after the ſame manner, if they are made of hewn ſtones. Every wall conſiſts of two walls or two parts, an outſide and an inſide wall; then the middle part is filled with lime and rubbiſh ſtones. Walls built of pebbles taken out of rivers, are built by no certain rule; for in both ſides of the wall, the ſtones muſt be laid as they will fit the places; and the inſide is filled with mortar and rubbiſh, called filling ſtones. And at every half yard high the wall is levelled, and a courſe of throughs is laid on, as I ſaid before. The form of a brick 60. wall, ſee Fig. 60.

When cellars are to be made; their foundations muſt be laid 7 feet below the firſt floor; for the top or ceiling muſt lie level with the ground on which the houſe ſtands, or very little higher. They ought to be entirely within ground, and to be ſituated on the north ſide of a houſe, for coolneſs. They are beſt arched over-head; for if they be covered above with a wooden floor, the wood ſoon decays, by the great dampneſs.

PROP. IV.

To ſhew the method of laying different ſorts of floors.

Ground floors are made various ways, as with bricks, flags, or marble. Square bricks from 6 to 12 inches, are made on purpoſe for flooring, and make a very good floor. Common bricks are alſo uſed for floors; they are laid length-ways croſs the houſe by a line, one courſe after another, but ſo that the joints at the ends may fall about the middle of each brick in the former courſe; this way they are laid flat, but they may be ſet edge-ways, and then they make a very ſtrong floor, but conſume twice as many bricks. Before they begin to lay them, they level the floor very exactly, and then

then lay a layer of ſand upon it, and ſpread it le- Fig.
vel, three or four inches thick. Flags are laid after the ſame manner; they muſt firſt be dreſſed plain on the upper ſide, and then made ſquare on all ſides, and they muſt be all made of the ſame breadth; and then they are laid courſe after courſe by a line, croſs the houſe; ſo that there may not fall joint upon joint.

Very good floors are laid with lime and gravel after this manner. After the lime has been ſlaked, and has laid ſome time, it is riddled through a wire riddle, to take all the lumps out of it. Then the ſharpeſt gravel that can be had muſt be got, and all the rougheſt part taken out by riddling; what remains conſiſts of very coarſe ſand, mixed with very ſmall ſtones. Put an equal quantity of this gravel, and lime together; and make it up like mortar. Then the houſe floor is to be levelled, and ſpread over with this mortar to the thickneſs of about 3 or 4 inches, and made very level. It muſt be left thus for a few days till it hardens; and when a perſon can walk on it without ſinking, he muſt take a beater made of a piece of flat board, and beat it all over very level; and when it begins to crack it muſt be beaten again. And thus it muſt be beaten ſix or eight times over, at the diſtance of 4 or 5 days, or a week, till it grows hard, and crack no more. But care muſt be taken that nothing comes upon it, till it be thoroughly dry. Inſtead of gravel ſome mix coals with the lime, and others ſmiddy gum (being the flakes that fly off hot iron) when it can be had. But the beſt thing is *ſparr* beat ſmall, which comes out of the lead mines.

Flooring with marble is the moſt beautiful as well as the moſt coſtly. It is laid in one entire ſtone before chimnies. And it is often laid of two colours placed chequerwiſe or ſquare, ſide by ſide;

or

Fig. or arraſs wiſe, laid angle to angle. And there are ſeveral forms contrived to lay pieces of different colours among one another, which appear very beautiful.

The cheapeſt flooring is that made with common rough ſtones, either gathered out of rivers, or raiſed in common quarries. This is proper for ſtables, courts, common high-ways, &c.

Floors of upper rooms are laid with deal boards or fir boards. In building a houſe, joiſts muſt be laid along to nail the boards to. Theſe joiſts muſt be made deeper than broad, for the greateſt part of the ſtrength lies in the depth, for the ſtrength increaſes as the ſquare of the depth, and only ſimply as the breadth. The depth may be 6 or 8 inches, or more if they are long, and be to carry a great weight. And they may be laid a foot or half a yard from one another, but not more than 2 feet. The boards muſt be plained and ſet to dry. Theſe boards ought to be 2 or 3 years old; or elſe they will pine, when laid in a floor; for they ought to be very dry when laid. And the broadeſt boards will pine the moſt; therefore to make good floors, ſo that the joints may not appear, they uſe narrow boards, called *battens*. Theſe boards when ſhot ſtreight are channelled on the edges, and latted; and driven cloſe together, and nailed down with brads or unheaded nails, which will ſink into the wood; and then the floor is plained ſmooth and even.

No dormants muſt be laid over the heads of doors or windows.

PROP. V.

To ſhew the proportion of the ſeveral parts of a room to one another.

An *entry* or *paſſage* ought to be ſo ſituated, as to have communication with all the other parts of the houſe;

houſe; it is a common place for people to ſtay in, who are waiting to negociate any buſineſs. This is the firſt part of the houſe that one enters into. It ſerves for a paſſage to and fro betwixt other rooms, and from the out-door into the houſe. Its dimenſions depend upon the other parts of the houſe. Fig.

The *hall*, this is a large room, which ſerves for feaſts, acting plays, or any grand entertainments. Theſe places ought to be very large, to entertain many perſons. The length of a hall ought at leaſt to be twice the breadth; or in large buildings, thrice the breadth. The height thereof ought to be $\frac{2}{3}$ the breadth, or it may be 16 or 18 feet in large buildings. They are ſometimes arched over, and then the height from the ground to the top of the inſide of the arch muſt be $\frac{5}{6}$ the breadth. Likewiſe the other low rooms may be arched.

Anti-chamber, is an outer-chamber in great houſes, where any perſons are to wait that go upon buſineſs. The breadth to the length ſhould be as 5 to 7, or a little more; and the height $\frac{2}{3}$ of the breadth, or ſome ſay $\frac{3}{4}$ the breadth. But if it be arched, take $\frac{5}{6}$ the breadth.

Anti-chambers and chambers ought to be ſo ſituated, that they may be on each ſide of the entry and hall; and nearly to anſwer alike on both ſides.

Of *chambers*, all the rooms above the firſt ſtory are called chambers, only thoſe at the top of the houſe are called *garrets*. The length of a chamber ought to be once and a half the breadth, at the moſt. And the height muſt be $\frac{3}{4}$ the breadth. Some ſay the length muſt be $1\frac{1}{5}$, or $1\frac{1}{6}$, or $1\frac{1}{7}$, or $1\frac{1}{8}$ the breadth; ſo this is all arbitrary. But if the chambers are arched, the height muſt be $\frac{5}{8}$ the breadth, meaſuring from the ground to the top of the arch within.

Chambers of the ſecond ſtory muſt be a twelfth part leſs in height than the chambers below. But if

Fig if they be arched, they muſt be a ſixth part leſs than thoſe below. In chambers of the third ſtory, the height muſt be $\frac{3}{4}$ the height of the ſecond chamber, or that below it. The dimenſions of garrets are left to the judgment and fancy of the builder; and muſt be contrived according to the particular occaſions and uſes they are deſigned for.

In the building of chambers, regard muſt be had to the ſituation of the chimney, the place of the bed, the places of the windows; ſo as to give them all ſuch ſituations as may beſt ſuit the purpoſe. The bigneſs and place of the paſſage muſt alſo be conſidered.

Galleries are a ſort of chambers made out of the houſe, commonly in the front; and are built upon pillars. They are long narrow rooms, made for walking, eating, and taking diverſion in. Their length muſt at leaſt be 5 times their breadth, but not more than 8 times. The lower galleries muſt be as high as the chambers of the firſt ſtory; ſo that one may walk on even ground out of the chamber into the gallery. Sometimes galleries are made ſo that one may go into them out of the court, without coming into the houſe; and then ſtairs are made to aſcend into them. And in this caſe the galleries may be lower than the firſt ſtory. Upper galleries muſt be as high as they are broad, if they be flat; but if arched, they muſt be a third, fourth, or fifth part more.

The low rooms, as the entry, hall, &c. are all upon a level, but are raiſed 2 or 3 feet higher than the court before the houſe; from which one aſcends into the entry by ſteps which may be 4, 5, or 6 inches high, but not more; and a foot or 15 inches in breadth.

PROP.

Fig.

PROP. VI.

To deſcribe the bigneſs of doors and windows, and the proportion of the parts.

The out doors of a large houſe muſt be proportioned to the bigneſs of the houſe. Let the doors and windows in any houſe be as few in number, and of no greater dimenſions than is neceſſary for free paſſage, and giving a due quantity of light. For all openings are ſo many weakenings. And let them not come too near the corners of the houſe, for then the corners will be weakened, which is the greateſt ſtrength of the houſe. Let the doors be over doors, and windows over windows, for it makes the work very weak to have a wall over a void place. Some ſay the doors ought to be placed ſo, that one may ſee through them all from one end of a houſe to the other, but this cannot always be done without loſing greater conveniencies. It makes ſtrong work to turn arches over the doors and windows, which will diſcharge a great deal of weight. Inner doors muſt be ſix feet high, and two feet and a half, or two feet nine inches broad. If doors be three feet wide, or more, they muſt be twice as high as broad.

Windows muſt be made ſo as to give a due quantity of light, for a room may have too much light as well as too little. A great room requires more light than a little room. And regard muſt be had to the ſituation of the place; if it be ſhaded, or towards the north, it requires the more light. In large houſes, the windows muſt be equal to one another in the ſame ſtory, and muſt ſtand in the ſame rank and order; and ſo that thoſe on the right may anſwer thoſe on the left; and likewiſe thoſe above to thoſe below.

In

Fig. In ſmall houſes the breadth of the windows within, may be $2\frac{1}{2}$ feet, or 3 feet, and the height $3\frac{1}{2}$ feet or 4 feet. But in large houſes, they may be 4 or 5 feet within the jaumbs, or in greater buildings 6 or 7 feet. And the height double the breadth, and ſometimes a third or fourth part more.

Windows of the ſecond ſtory muſt have the ſame breadth, but they muſt be lower by a twelfth part, but ſome ſay they muſt be lower by a third part. And the windows in the third ſtory muſt be a fourth part lower than thoſe in the ſecond.

The height of the window ſoles muſt be 2 feet 8 inches, or not above 3 feet. In ſmall houſes the window-frames muſt be $2\frac{1}{2}$ or 3 inches thick, which muſt be of oak. But in large houſes they are made 4 or 5 inches thick. When a window is wide, each ſhutter muſt open in the middle with joints. Door poſts of oak ought to be 6 or 8 inches ſquare in large houſes, and 4 or 5 inches in ſmall houſes.

Pilaſters of doors and windows muſt be $\frac{1}{5}$ or $\frac{1}{6}$ part of the aperture, and muſt project $\frac{1}{6}$ part of the thickneſs.

There are ſeveral ornaments for doors and windows, as the architrave, frize, and cornice. The architrave ought to be as thick as the pilaſter; and the frize above that muſt be $\frac{1}{4}$ the height of the architrave, and the cornice at the top muſt be $1\frac{1}{4}$ the height of the architrave. Their ornaments and mouldings are made according to that order of columns, which the builder thinks proper.

PROP. VII.

To give an account of the conſtruction of chimnies, and the methods attempted to cure their ſmoaking.

In ordinary houſes the breadth of a chimney may be two feet, the height 3 feet 8 inches, or 2 feet 1

of

or 2 inches above the bars; and the depth back- Fig.
wards 15 inches within. The top or mantle-tree to project 5 inches over the jaumbs, and the bars ſet 10 inches further out than the jaumbs: theſe are convenient meaſures. But in large buildings, hall chimnies muſt be 6 or 7 feet wide, and in great buildings 8 feet between the jaumbs; their height 4 or 5 feet, and depth backward 2 or 3 feet. Chamber chimnies muſt be 5, 6, or 7 feet wide, and 4 or 4½ feet high, and 2 or 2½ feet deep. Leſſer rooms may have the chimnies 4 or 5 feet broad, 4 feet high, and 2 feet deep.

If it can be done, make the funnels in the thickneſs of the walls, which will ſave a deal of room in the houſe; and theſe funnels muſt be carried through the roof 3, 4, or 5 feet, to convey the ſmoak away. They muſt be made of a proper wideneſs; for if they are too wide, the wind will drive the ſmoak back into the room; and if they be too narrow, the ſmoak cannot have free paſſage. No chimney-ſhaft ſhould be ſtraiter than 10 inches; chamber chimnies may be 10 inches deep and 15 wide; and hall and kitchen chimnies 15 inches both ways. The ſhafts muſt be carried up very ſtreight. It often happens, that making the chimney-piece too high cauſes the room to ſmoak; and therefore it is better to make them lower than theſe ſtated rules.

There are ſeveral cauſes for the ſmoaking of chimnies. If a houſe is ſo ſituated, that it be overtopped by other houſes, or by trees, hills, &c. they are apt to ſmoak, for the goal that comes behind theſe obſtacles, will ſtrike down the chimney; except theſe obſtacles be very high above your chimney, for then the wind flies quite over, and does no damage. But if it happens that the wind be contrary, and blow againſt another houſe higher than yours, and beyond yours, and very near it,

Fig. it is hardly poſſible yours ſhould avoid ſmoaking! But there may be ſome fault in the building, by making the funnel too wide or too narrow.

Several ways have been tried to cure ſmoaky chimnies, which ſometimes ſucceed. One way is to obſerve what wind makes the houſe ſmoak; and on that ſide, make a hole in the top of the chimney about the middle, in which fix a flat ſlate, ſloaping upwards. The wind blowing in there, will go out of the chimney top, and the ſmoak will follow.

Another way is to place an eolipile in the chimney, where it can be heated, and the vapour going out of it will drive the ſmoak up the chimney.

Sometimes a vane is put at the top of the chimney, which carries about a broad piece of tin lying aſlant from the wind. This tin being always turned againſt the wind, ſhelters the ſmoak, which goes out freely.

Covering the top of the chimney with a broad ſtone, ſet 5 or 6 inches above it, being ſupported at the corners by four pillars, will ſometimes prevent ſmoaking, when the wind tends directly down the chimney. And ſometimes covering the top with two ſlates lying aſlope like the roof of a houſe, leaving two oppoſite ſides open, will cure it.

Sometimes the doors of a room are badly ſituated, and the ſhifting of them has cured the ſmoaking.

When the fire-place is too wide, the chimney is apt to ſmoak; and then it is prevented by building two cheeks within it, to make the paſſage narrower. For the ſtraiter the paſſage, the ſwifter the freſh air will ruſh in towards the fire, which being rarified, will naturally aſcend up the chimney, and carry the ſmoak with it.

When the funnel of a chimney is made too narrow, there is no remedy for the ſmoaking, but to keep a very ſmall fire.

When

When the chimney is in an out-wall; making a hole in the fire-place, through the wall, to let a freſh ſupply of air come in, often cures it. For the cloſeneſs of a room is often the occaſion of ſmoaking, for if a freſh ſupply of air does not come in, to make up for what is expended in carrying out the ſmoak, one may be ſure the ſmoak cannot paſs off, but will fill the houſe. Fig.

Sometimes the ſmoaking may be cured by making two oppoſite ſides of the chimney top, higher than the other two, by a courſe or two of bricks; which is to be done according to the poſition of the wind. For in almoſt all caſes the wind has a hand in the ſmoaking of chimnies. A funnel built behind the fire with a mouth a foot ſquare, ſometimes cures the ſmoaking.

PROP. VIII.

To deſcribe the different methods of making ſtairs and ſtair-caſes.

A ſtair-caſe ought to be ſo placed, that when the ſtairs are made, the communication with all the chambers may be eaſy and convenient; and the room in it ought to be proportioned to the reſt of the houſe; it muſt have a due quantity of light, therefore if it have but one window, it muſt be placed in the middle, to enlighten the whole. There ought to be at leaſt one half ſpace before one come at the landing-place on the top. For where there is but one flight of ſtairs, that is, where they go in one direction to the top, without any reſting-places, it is very dangerous going down. For if one happens to ſlip or fall, they muſt go to the bottom at once, there being no turn, or any thing to ſtop one. Yet ſuch ſtairs are forced to be made ſometimes into garrets where room is ſcarce.

Fig. The landing-place muſt be large enough to lead to all the chambers. In ſmall houſes the height of each ſtep may be 8 inches, or not above 9; and the breadth a foot, or not leſs than 9 inches, and the length three feet. But in large houſes the height of a ſtep muſt not be more than 6 inches, nor leſs than 4, their breadth not leſs than 12 inches, nor more than 16. And the length 4, 5, or 6 feet. But in ſome very great buildings they are made longer.

As to the form, there are various kinds, denominated in general by *ſtreight ſtairs* and *winding ſtairs*. The ſtreight ſtairs always fly, but never wind; and are of ſeveral kinds, as

Direct fliers, theſe go directly from one floor to another, without turning to the right or left, ſeldom uſed but for cellars or garrets.

Square fliers, theſe fly round the ſides of a ſquare newel; and at every corner of the newel there is a ſquare half-ſpace, after which the next flight goes off at right angles to the former. A *newel* is the upright poſt, about which a pair of ſtairs turns, being placed in the middle of the ſtair-caſe. Sometimes the newel is a ſolid pillar, and ſometimes an open ſpace.

Triangular fliers, theſe fly about the ſides of a triangular newel. So every flight goes off from the laſt in an angle of 120 degrees, or 60°, reckoning backwards; and at each angle is a half pace. All ſtairs in general that go directly forward, are called *fliers*.

French fliers, are thoſe that fly firſt directly forward to the half-ſpace; and then directly back again to the top, or to another half-ſpace; and the next flight (if there be any more) goes over the firſt, &c. Theſe ſtairs are in common uſe.

Winding ſtairs, are thoſe that never go ſtreight, but continually change the direction. Of theſe

ſome

ſome wind round about a circle, ſome about an el- Fig.
lipſis, others about a ſquare or a triangle. And each ſort may either go about a ſolid newel or an open one. In all winding ſtairs, the ſteps are broader at one end than the other.

Circular winding ſtairs, are thoſe which go about a ſolid newel; here the edge of each ſtep is a right line pointing to the center of the newel. They are often found in church-ſteeples and cathedrals. The ſteps are of the ſame form, if the newel be open, as in the monument at *London*. In ſome ſorts of theſe, the foreſide of each ſtep is circular or elliptical, according to the different fancies of people.

All kinds of winding ſtairs take up leſs room than ſquare ones, or any others.

Where there is much room, the length of a ſtep muſt be equal to the diameter of the newel. In other caſes make the diameter of the newel $\frac{1}{4}$, $\frac{1}{5}$, or $\frac{1}{6}$ the diameter of the whole ſtair-caſe, according as the ſtair-caſe is great or ſmall. If the newel be open, make the newel half the diameter of the ſtair-caſe, to be more enlightened. In very narrow places, theſe ſtairs may be made without a newel.

Elliptical winding ſtairs, are divided like the round; here all the windows and doors are in the middle and head of the oval. The newel here is alſo an ellipſis.

Stairs may alſo wind round a ſquare or a triangle, and many other ways may be invented.

Double winding ſtairs are ſo contrived, that two perſons, one aſcending and the other deſcending, ſhall not come at one another. The foot of one of theſe ſtair-caſes is oppoſite to the foot of the other. And they make their aſcent on oppoſite ſides. The newel in the middle is hollow to convey light, and very large.

Fig. *Quadruple winding ſtairs*; there are four ſtair-caſes which have four entrances, and go up one over the other, and being made in the middle of a building, they lead to four chambers; ſo that the inhabitants of one need not come upon the ſtairs of the others. This muſt have a very large newel, which is open, to give light. They all ſee one another go up and down, without hindering one another.

In ſquare ſtair-caſes, when you are forced to place ſteps in the angles for want of room; you cannot place more than three ſteps at an angle, to wind; when there is but 6 or 7 feet in breadth. For 8 feet broad, take 4 ſteps at an angle. And for 9 or 10 feet width, take 5 ſteps. Theſe ſorts are called *mixed ſtairs*, *dog-legged ſtairs*, and *fliers and winders*.

There are a ſort of ſtone-ſtairs, going up into towers, which wind; and every ſtep has one end in the wall, and is ſupported by the lower ſtep, where the foreſide of the upper ſtep lies upon the backſide of the under ſtep, all the way.

The landing-place of any ſtairs, or the ſpace between the wall and the ſteps, muſt be in breadth, the length of a ſtep, and at leaſt a fourth part more.

To find the number of ſtairs.

Having pitched upon the height of a ſtep; divide the height of the room (in inches) by the height of a ſtep; the quotient is the number of ſteps.

To find the breadth of a ſtep, divide the width or ſpace (meaſured on the ground) by the number of ſtairs; the quotient ſhews the breadth of a ſtep.

SCHOL.

To find the twiſt of a rail in a ſtair-caſe. Suppoſe the ſection of the rail to be a ſquare or rectangle, and the newel to be circular, and that the ſtairs

ſtairs turn round in a circle. Let *a* be the height of a ſtep, *b* the breadth; *r* the radius of the newel, or the diſtance from the center to the inſide of the rail, R the diſtance to the outſide. Then the radius of curvature of any part of the rail on the inſide will be $\frac{aa+bb}{bb}r$; on the outſide $\frac{aa+bb}{bb}$ R. Fig.

Let ABC be a ſtep, AB (*b*) the breadth, BC (*a*) the height, A = degrees in the angle BAC, *d* = hypothenuſe AC, *c* = 3.1416. Then the twiſt in a quadrant will be A, and the twiſt in every foot of length of the rail is $\frac{2b}{cdr}$ A, in degrees; taken on the inſide. But it is the ſame in any ſide, as the Section is every where the ſame. And the twiſting lies from the newel, in the aſcent. 61.

The upper ſide in the aſcent is concave, and therefore leaves the plain (of an ellipſis) which touches it at firſt. But the deviation is ſo ſmall, that it may be neglected. The length of a quadrant of the rail is $\frac{dcr}{2b}$. And the deviation in $\frac{1}{100}$ part of this length is but $\frac{.000645ar}{d}$. The practice may be thus, take a piece of timber as *d*DH*b*, and with a line of $\frac{dd}{bb}r$ feet for the radius, deſcribe with chalk the arch *bfd* on the inſide, and with the radius $\frac{dd}{bb}$ R, the arch HFD; the part between theſe arches is the rail. Then ſince the twiſt affects every ſide of the rail alike; lay it down on a plain floor, ſo as the upper ſurface at D be parallel to the horizon, and make DE, EF, FG, GH, &c. each a foot; the length of a quadrant of the 62.

Fig. 62. rail being $\frac{dcr}{2b}$; and drawing E*e*, F*f*, G*g*, H*h*, &c. perpendicular to the ſide DH; place an inſtrument to the angles, firſt upon D*d* which will be level; and then upon E*e*, F*f*, &c. Now the upper ſide is to be ſo formed or cut away, that the inſtrument placed upon E*e* will lean to the right hand $\frac{b}{1.57dr}$ A; upon F*f*, $\frac{2b}{1.57dr}$ A; upon G*g*, $\frac{3b}{1.57dr}$ A, and ſo on. And the other ſides muſt be cut ſquare to it, in every place. And the top ſide muſt be a very little concave.

PROP. IX.

To deſcribe the ſeveral methods of roofing.

The inſides of roofs are made of timber; and on the outſide they are covered with tiles, ſlates, lead, ſtraw or ling; and the form of the roof will be different for each of theſe. There are two ſorts of roofs, *plain roofs* and *hip roofs*; the plain roofs have gable ends, and the hip roofs have none.

The wood work of a plain roof is made thus; if a room be above 12 feet long, there is placed in the middle of it a pair of ſiles or *ſuſpars*, called the *principal rafters*, on the top is laid the *riggin-tree*, which lies horizontally with its ends on the tops of the gable-ends of the room, and its middle on the ſuſpars. On the ſides are laid the *ribs*, or *purlins*, one or two on a ſide, according to the wideneſs of the room; theſe are to nail the ſpars or rafters to, which are to be laid all along, at about a foot and half diſtance. Then the *lats* are nailed croſs the ſpars at about 11 inches diſtance for tiles. In ſhort rooms there is no occaſion for ſuſpars,

ſuſpars. Suſpars ought not to lie over the heads of windows or doors. Fig.

For the *pitch*; let AB be the breadth of the houſe, make AC and CB $= \frac{3}{4}$ AB, to meet at C, then that is called the *true* pitch. And when the gable-end ACB is built; two ſpars or rafters AC and BC, of a juſt length, are nailed together at the top C, and ſet upon the walls at A and B; and the gable-end is built by theſe. Here the perp. CD is $= \frac{AB}{4}\sqrt{5}$. But it is now very common to make roofs under pitch, ſo as the angle ACB may be a right angle; and then the perpendicular CD is equal to AD, or half AB, and it is called the ſquare pitch. No roof ſhould be made lower than ſquare, for then it will rain in. Here CC is the riggin-tree; FF, FF the ribs. In the ſquare pitch AC is a mean proportional between AD and AB, and therefore $AC = AB \times \frac{\sqrt{2}}{2} = \frac{AB}{\sqrt{2}}$. In ſome caſes they make the pitch, ſo that AC = AB, which is very high. 63.

For hip roofs. Let LA be the top of the ſide wall (or you may lay it out on the ground), CN the riggin-tree; AB the breadth of the houſe, AD = DB, and CD a perpendicular, then the plane ACB will be perpendicular to the horizon. And ACB will repreſent a gable-end, inſtead of which the hip-roof is to be made, and the hips muſt meet at the point C. Produce LA to G, ſo that AG = AD, and draw GK parallel to AB; and make the two ſquares GADH, and HDBK. Then taking AC of a proper length, GC and KC will be the two hips, whoſe length will be $\sqrt{AC^2 + AG^2}$, or $\sqrt{AC^2 + \frac{1}{4}AB^2}$. AB is a ſtrong *dormant*, or *ſummer*, going croſs the houſe, GD, KD the *dragon beams*, for the hips to ſtand on, 64.

Fig. on, being framed into the beam AB at D. The ribs are to lie along upon NL, CA, and CG; and other croſs ribs are to lie upon CG and CK, upon all theſe the ſpars are to be nailed. The length of CH is equal to CA. The *back* of the hip is the angle made by the two planes CGLN and CGK. The planes CGD, CKD are perpendicular to the horizon, both paſſing through the perpendicular CD.

The practice is much the ſame for *bevel* roofs. A *bevel houſe* or roof is one where the ends are not perpendicular to the ſides. As ſuppoſe the angle LAB is an oblique angle; then GADH and KBDH are two rhombus's. And the length of $GC = \sqrt{GD^2 + CD^2}$, and $KC = \sqrt{KD^2 + CD^2}$.

Tiling. Of tiles there are various ſorts, as common or pan tiles, plain tiles, riggin tiles, gutter tiles, Flemiſh tiles, hip tiles, Dutch tiles, &c. In covering a houſe with pan tiles, one muſt be very exact in the gage for placing the lats. For if the gage be too ſhort, the tiles will want lap on the ſides, which is a great fault. And if it be too long, they will want lap on the tops, but this is not ſo material. Flat gutters muſt be laid with lead, and ſo ſhould all gutters, but ſloaping gutters are often laid with gutter tiles, and ſometimes with common tiles. Tiles laid on a houſe-ſide are laid without lime, and pointed on the inſide with lime mixt with hair. The riggin is covered with riggin tiles, made hollow on one ſide, and broader than the other ſort. But workmen often make ſhift with common tiles, cutting off the roll, theſe are laid in mortar. A ſhort length ſometimes happens at the bottom of the roof; and then the workmen cuts away the ſhoulder of the tiles, till they will fit into their places. In hip roofs, the hips are covered with hip tiles, which are much like riggin tiles. Hip tiles have a hole in one end,

to

to nail them on by. Some houſes are covered with plain tiles, which hang by pins, in the nature of ſlates. Fig.

Dutch tiles are only uſed inſtead of chimney corner ſtones.

In flat gutters that are laid with lead, take care that the gutter boards have a good deſcent. If one ſheet of lead does not reach the length, the plumbers ſolder another to it, which often breaks in the joint, and lets in water. A better way is to put the end of the ſecond ſheet under the end of the firſt, making a proper deſcent for it in that place.

Slating. Slates are flat ſtones like flags, raiſed out of a quarry, and are an inch or two thick, according to the breadth; they are dreſſed into the form of a parallelogram, of different ſizes from 3 feet to 1 foot. The timber here muſt be ſtronger than for tiles, and the lats muſt be oak, and laid wider at the bottom than top of the roof, for the biggeſt ſlates come on firſt. The ſlates have one or two holes in the top. The roof is made hollow, or pretty ſharp at the top; otherwiſe the ſlates would not lie, but ride upon the heads of one another. Tiles are laid in rows upward from the bottom to the top of the roof; but ſlates are laid in horizontal rows, beginning at the bottom with courſes of the largeſt ſlates. They are all laid in moſs, and hang on the lats, by bones or oak pins put through the holes. As you riſe higher, the courſes decreaſe gradually to the top; they muſt not be laid joint upon joint, the ſlates in the top courſe are toothed or notched on both ſides, and the riggin ſlates of one ſide of the roof fixt into thoſe on the other ſide, ſo that they cannot blow off. They are pointed with mortar for 2 or 3 courſes at top and bottom. The roof is 3 tile thick at every point, if right laid. To try the goodneſs of ſlates, hang them by an edge in your fingers,

Fig. fingers, and ſtrike againſt them with a ſtone or with a finger; if they ring they are good; but if they chatter, they are ſoft, or cracked in ſome place, and ſo are uſeleſs. The ſame way tiles are tried.

Leading. When houſes or churches are covered with lead, which is very chargeable; large dormans of oak are laid croſs the building at 9 or 10 feet diſtance, made higheſt in the middle; and on theſe, as many ribs as are ſufficient, and ſpars nailed to theſe, all of oak; and inſtead of lats, thin oaken boards are laid cloſe, upon the ſpars, and nailed down, like a floor. Then the ſheets of lead are laid cloſe by one another, reaching from bottom to top. And the edges are turned over one another, and beat down round, incloſing one another, ſo that no rain can get in at the joint. When a ſheet is too ſhort, another ſheet is laid on below, and its end put 3 or 4 inches under the end of the firſt. They muſt be nailed down in ſeveral places, with ſhort nails with broad heads.

Straw is another covering. It is laid by a pool of water for ſome days, and is conſtantly watered. Then it is drawn into ſheaves, and led away, and muſt lie for a month at leaſt, before it is thatched; when it is thatched, there is firſt made a deal of bundles of dry ſtraw, which are laid all the way croſs the ſpars, and tied down with ropes of ſtraw, going round the ribs and lats. Then the *theaker* begins to thatch at the bottom, and carries up a courſe of thatch, two feet wide, to the top; and at the top lays ſome mortar upon it. The manner of thatching is this, the workman has a ſhaff of thatch brought up; he takes a part of it and makes a wiſp of it, tying the top faſt together with part of the ſtraw, to keep it together; then he thruſts his inſtrument (the ſprittle) into the dry bundles of ſtraw (if it is a new houſe, or into the old covering, if it is an old one), to make a hole, into which

which he thruſts the wiſp with the head foremoſt, Fig.
with the ſprittle, as far as he can get it. Then he puts in another by the ſide of it, and another by the ſide of that as cloſe as he can ſqueeze them in, till he has got the breadth of his courſe; then he works in another croſs row of wiſps, and above that another, and ſo on till he comes at the top. The ladder he ſtands on to work, terminates the breadth of each courſe. As ſoon as each courſe is thatched, he takes a ſwitching knife and ſtrikes off the top of the courſe, cutting it ſmooth all the way down, and cuts the eaves ſtreight at the bottom. In ſome places they thatch with reeds inſtead of ſtraw, which laſts many years. Thatching with ſtraw requires a ſharper roof than tiles.

Ling. Here the pitch muſt be very great, and very ſtrong timber to bear a coat of ling. Ling is thatched by only laying one courſe upon another, horizontally, all the length of the houſe; it muſt be laid ſloping, with the top outward, this is no more than making a hay ſtack; but it ought to be tied on here and there with ropes, or faſtened with ſpelks. To make the riggin, a number of ſods is pared of a proper length, and laid croſs, all along the riggin, and ſpelked or pinned down. One of theſe roofs requires a great deal of ling, and riſes very high.

Lead is the beſt cover, but is immenſely dear; but for common houſes, ſlates are quite the beſt covering; for they are neither hot in ſummer nor cold in winter; but tiles are both. Neither are they ſo ſubject to rain in as tiles. A ſlate roof is eaſily repaired, they only require ſtopping with moſs once in 5 or 6 years. A thatched houſe is far warmer than a tiled houſe; but then thatch does not laſt long; and is ſubject to be blown off in high winds. Tiled roofs are ſo cold in winter, that moſt people are obliged to ceil them on the inſide.

And

Fig. And many people ceil the top of a room, on the underſide of the joiſts, and ſometimes on the boards. In a garret there are joiſts laid on purpoſe, called *ceiling joiſts*. Ceiling of rooms makes them more lightſome, and leſſens the noiſe above head. To ceil a room, there muſt be ceiling lats nailed all along on the underſide of the joiſts, as rank as may be, and theſe are to be plaiſtered over with lime, made up with hair; as all plaiſtering is made.

PROP. X.

To explain the ſeveral orders of architecture.

In the making of columns, there are five different forms and proportions, called the five orders, the *Tuſcan*, *Doric*, *Ionic*, *Corinthian* and *Compoſite*. A column conſiſts of three parts, the *pedeſtal*, the *column or body*, and the *entablement*. The parts of the pedeſtal are the *baſe*, *die*, and *cornice*. The parts of a column are the *baſe*, *ſhaft*, and *capital*. The parts of the entablement are the *architrave*, *frieze*, and *cornice*. Theſe and their parts are called *ornaments* or *mouldings*. All their parts are meaſured, and ſet off by *modules*. A *module* is the diameter of a column taken at the baſe, and is divided into 60 parts or minutes. ABC is a column,
65. A the pedeſtal, B the ſhaft, C the capital, above is the entablement. Every column muſt be ſmaller towards the top than at bottom, taking the example from the boles of trees that taper all the way up. A column is ſaid to be more beautiful, if it ſwell a little about the middle, or rather at about a third part of its height.

The capitals of the Tuſcan and Doric are plain, that of the Ionic has a *ſcrol* or *volute* upon it; the Corinthian and Compoſite are adorned with flowers, and the leaves of trees. The height of the entablement,

blement, or of the architrave, frieze and cornish, Fig. is $\frac{1}{4}$ the height of the column, in the Tuscan and Doric; and $\frac{1}{5}$ the height in the Ionic, Corinthian and Composite.

An account of the ornaments which are to be found in some or other of the orders; going upwards from the top of the column.

The *capital* is the head of the column; near the top of the column goes a round list.

Architrave, the moulding above the capital, made upon a stone or beam of wood which may go over several columns.

Annulet, *fillet* or *list*, a narrow flat moulding which goes round the column.

Astragal, a little roundish moulding mostly at the top of a column.

Frieze, a large flat member above the architrave.

Echine, or *ovola*, a part above the frieze, a quarter round on the lower part.

Voluta, an ornament that represents the barks of trees twisted and wreathed, in the Corinthian order.

Abacus, a square table on the top of a pillar.

Cimace, *scima recta*, or *ogee*, a member consisting of a round below, and a hollow above, like an S.

Scima reversa, an ogee turned upside down.

Fasciæ, broad lists, belts, or fillets, commonly made in the architrave.

Trygliffe, part of a frieze.

Cavetto, or *scatia*, a hollow moulding.

Crown, the finishing part of an ornament.

At the bottom of the column are,

Annulets, or flat rings.

Astragal, as before.

Scotia, or *cavetto*.

Torus, a round member going about the base.

Plinth, a square part, the lowest part of the base.

Cimace,

Fig. *Cimace*, as above.

Die, a ſquare part included between the baſe and cornice.

A Table ſhewing the length of the principal parts in the five orders, in modules and parts.

	body		pedeſt.		baſe		capital		archit.	frieze	corn.	
	m.	*p.*	*m.*	*p.*	*m.*	*p.*	*m.*	*p.*				
Tuſcan	6	15	2	0	0	30		30	30	30	35	
Doric	7	30	2	35		30		30	30	45	40	
Ionic	8	0	2	50		35		22	36	32	48	
Corinth.	8	10	3	0		30	1	10	37	35	1	0
Compoſ.	8	22	3	25		30	1	8	45	42	51	

The diameters of the columns at the top, are to be as follows, the *Tuſcan* $\frac{3}{4}$ the diameter at bottom. The *Doric* $\frac{4}{5}$; the *Ionic* $\frac{5}{6}$; the *Corinthian* $\frac{6}{7}$; the *Compoſite* $\frac{7}{8}$, the diameter below.

The Tuſcan column is the ſtrongeſt of all the orders; the next the *Doric*, then the *Ionic*, the *Corinthian*, the *Compoſite*. And in building, the ſtrongeſt muſt be ſet loweſt. Therefore the *Doric* is placed under the *Ionic*; the *Ionic* under the *Corinthian*, &c. The Tuſcan is ſeldom uſed but under ground.

The inter-columns, that is, the ſpaces between the columns, are thus regulated. In the *Tuſcan*, that ſpace muſt be 4 diameters at the baſe, in the *Doric* 3; in the *Ionic* 2; the *Corinthian* $2\frac{1}{4}$, *Compoſite* $1\frac{1}{2}$ diameters below. The baſe and capital of any column ought to project, beyond the body $\frac{1}{5}$ or $\frac{1}{6}$ the diameter at bottom.

Columns placed in any front of a houſe muſt be all of the ſame ſort, of the ſame bigneſs and height; and at the ſame diſtance, except where there are doors, and there they muſt be wider, as in the middle of the front, where the main door is generally placed. Pillars are frequently placed under piazzas (or P. Hes) and cloyſters.

Pilaſters,

Pilaſters, or half pillars, are thoſe that uſually Fig.
ſtand behind columns for arches to ſtand on; the whole appear like double or clud columns. They are often made to jut out of a wall, about a third part of their diameter. When they adjoin to a column, they have the very ſame dimenſions, capital and baſe with the column, according to its particular order.

The height of the pedeſtal of any column muſt be a third part of the whole column; or more particularly, $\frac{2}{9}$ for the *Tuſcan*; $\frac{1}{4}$ for the *Doric*; $\frac{3}{14}$ for the *Ionic*; $\frac{4}{9}$ for the *Corinthian*; and $\frac{3}{13}$ for the *Compoſite*. The height of the capital $\frac{1}{2}$ diameter, in the Tuſcan, Doric, and Ionic. And 1 diameter in the Corinthian and Compoſite.

The *Perſian order* of architecture, is to have the figures of men and women inſtead of columns, to ſupport any building. There is alſo the *Attic* order, where ſmall pillars, &c. are placed upon greater works; and the *Gothic* order, which is a ſtrong and coarſe manner of building; and the works are often carved with the figures of thiſtles, coleworts, &c. And Gothic arches are thoſe that conſiſt of two arches, making an angle at the top.

Arches are often made to ſtand upon pillars as in all churches; and in bridges, where the piers are made exceeding ſtrong, to reſiſt the water.

Columns are often made *fluted*, that is, with ſemicircular hollows running along the body of the column, from the baſe to the top, 24 in number. In the *Doric*, there is no ſpace between; in the reſt, a liſt runs between the flutes.

A draught of the ſeveral parts or ornaments will 66.
give a better notion, than a verbal deſcription; as follows. A*a* ſcima recta, B*b* ſcima inverſa, C*c* the crown, D*d* torus, E*e* cavetto, all theſe belong to the cornice. Then F*f* is the freeze. G*g* the architrave. In the capital, H*b* the cimatium, I*i* the

Fig. abacus, K*k* echinus, L*l* aſtragal. M the body of the column.

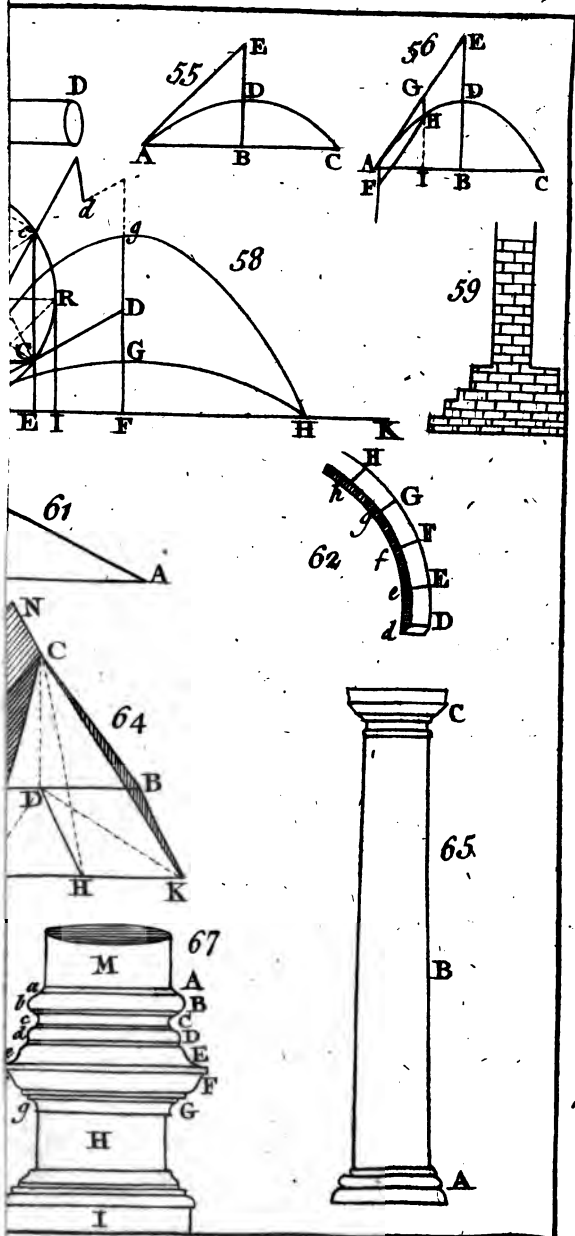
67. In the baſe, M the body of the column, A*a* liſt, B*b* torus, C*c* cavetto, D*d* torus, E*e* plinth. In the pedeſtal, F*f* cimacium and liſts, G*g* cavetto, H the ſquare, I baſe.

I ſhall not trouble the reader with the meaſures of theſe parts, as almoſt all authors differ therein from one another; for as they are all deſigned for beauty and ornament, different people have different notions of what is beautiful; ſome people thinking them prettier in one form and ſome in another. And the ſame obſervation may be made concerning the ſtated lengths and thickneſs of columns, for all theſe canonical proportions are merely arbitrary, and given them for the ſake of beauty. But true mechanical beauty conſiſts in giving a due proportion of ſtrength to every thing, and to every part of it. And the bigneſs of columns ought to depend upon the hardneſs and ſtrength of the matter they are made of. If a ſolid pillar of braſs or iron was to be made, of the ſtated height of any of theſe orders deſigned for ſtone, it is evident, it need not have its diameter ſo great, to bear the ſame weight. And if a column was to be made of the ſofteſt ſort of ſtone, its diameter would require to be made much greater, or elſe it would be cruſhed to pieces with the incumbent weight. So that it is abſolutely neceſſary that the dimenſions muſt depend upon the ſtrength and goodneſs of the materials; come what will of its beauty, which is nothing but fancy.

SCHOL.

It may not be amiſs here to ſhew the inveſtigation of two curious mechanical Problems, concerning the ſtrength and preſſure of walls and pillars.

Prob.



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Prob. 1. *To find the nature of the curve* AEB *or* AFC, *in the form of which, a wall of a given thickneſs being built, it ſhall be of equal ſtrength every where to ſuſtain the weight; or ſhall not be ſooner cruſhed to pieces in one part than another.* Fig. 68.

Let AD $= x$, DE $= y$, the thickneſs $= 1$; then the fluxion of the figure is $y\dot{x}$, and the weight is as $Fl : y\dot{x}$. Alſo the ſtrength at D is as DE or y; therefore $Fl : y\dot{x}$ is as y, or $Fl : y\dot{x} = ry$, and taking the fluxions $y\dot{x} = r\dot{y}$, and $\dot{x} = \frac{r\dot{y}}{y}$, and $x = r$ log. y, which denotes the logarithmic curve, whoſe aſymptote is AL. Therefore if a wall ABC be built, whoſe two ſides AB, AC, are in the form of the logarithmic curve, of infinite length AL, it will no ſooner be cruſhed to pieces in one part than another.

Cor. *If the wall* EGHF *be made upon* EF, *whoſe height* EG *is the ſubtangent to the curve; and the part* AEF *taken away, the part* BEFC *will ſtill be of equal ſtrength every where.*

For (Cor. 1. Prop. IV. log. curve) the area AEF $=$ GEFH, and the weights equal.

Prob. 2. *To find the nature of the curve* AEB, *that being turned round its axis* AL, *ſhall generate a ſolid pillar, that ſhall be of equal ſtrength in every part.*

Let AD $= x$, DE $= y$, $c = 3.1416$; then the fluxion of the ſolid is $cyy\dot{x}$, and the ſolidity or weight, as $Fl : cyy\dot{x}$. Alſo the ſtrength at D is as cyy; therefore by the queſtion, $Fl : cyy\dot{x}$ is as cyy, or $Fl : cyy\dot{x} = rcyy$, in fluxions $cyy\dot{x} = 2rcy\dot{y}$, and $\dot{x} = \frac{2r\dot{y}}{y}$, and $x = 2r$ log. y, whence AEB is a logarithmic curve, as before, and its aſymptote AL. Therefore a ſolid pillar, in form of the logarithmic

Fig. mic curve of infinite length, will no ſooner be cruſhed to pieces in one part than another.

Cor. *If* EG = *half the ſubtangent, and* EGHF *be a cylinder placed upon* EF, *inſtead of the ſolid* AEF. *Then the part* BEFG *of the pillar will ſtill be every where of equal ſtrength.*

For (Prop. V. log. curve) the ſolid AEF = cylinder EGHF, and therefore their weights are equal.

And hence it alſo follows, that if the wall or pillar be made of any glutinous or tenacious matter, and turned upſide down, and ſuſpended at the thick end in the air; its weight will no ſooner pull it in pieces in one part than in another.

PROP. XI.

To ſhew the ſituation and magnitude of gardens, courts, gates, woods, &c.

No houſe ſhould be built too far from the ſtreet, that is, too far back, if it ſtand in a town; for it cauſes a great deal of labour going back and forward to the houſe, and nothing can be ſeen in the town. And when it happens that a houſe has been injudiciouſly placed too far back, this fault may in part be remedied, by the loſs of a little ground; and that is, by taking away the ſegment of a circle from the yard or court, and let it lie to the ſtreet.

An area or court before a houſe ought to be as broad as the houſe, and muſt be a ſquare or rectangle; if a rectangle, the length muſt be the breadth and half, but not more. But towards the ſtreet, it may ſometimes be terminated by the arch of a circle or ellipſis. And no trees ought to be placed in it, but only ſhrubs, which one can ſee over.

over. Gravel walks muſt be laid, ſtreight to the door, and round about, and borders of flowers planted to beautify it. The reſt to be graſs. Fig.

Gardens muſt be made large enough to grow ſuch fruits and vegetables as are ſufficient for the family. A garden ought to join to the ſouth ſide, or the warmeſt ſide of a houſe, which will be very pleaſant and beautiful for moſt part of the year; being placed in a warm ſituation, they will be the better defended from the cold blaſts of the ſpring, that oftentimes deſtroy all fruit. The north and north-eaſt winds do endleſs miſchief, and muſt be guarded againſt. An orchard may ſtand at a diſtance from a houſe, but if it have a warm ſituation it will be better. It is commonly placed on the backſide of a building, whatever way that happens to be.

Large trees may be planted at ſome diſtance from the houſe, to defend it from cold and ſtormy weather. But by no means let them be placed ſo as to hinder a good proſpect; and a proſpect is the better, as it is the more extenſive, and conſiſts of more variety of objects. Hills and dales, and a mixture of land and water are very pretty, as are woods and groves, at a diſtance. A houſe is beſt defended by trees, when they are placed towards the ſouth-weſt or north-eaſt; for the ſtrongeſt winds come from the former, and the cold weather from the latter; and all theſe circumſtances ought to be conſidered.

Gates for foot people to paſs through ſhould not be leſs than four feet wide; nor leſs than eight for carriages. And in large buildings, they may be 10 or 12 feet wide, but not more. The principal gate muſt face the main door of the houſe; a fair entrance gives a great grace to a building. Some ſay the height of a gate ſhould be once and

Fig. a half or twice the width, which in large gates is certainly more than is needful. The gate ſhould be loweſt in the middle; and the top, of a circular form. Ornaments are often placed over the tops of gates and out doors, which makes them look very pretty.

Many more particulars ought to be taken notice of, which, for ſcarcity of room, I am forced to omit. But theſe I have treated on are the principal.

ART.

Fig.

ART. XV.

MUSIC.

MY deſign here is to treat of this delightful art, ſo far as it depends on mathematical and philoſophical principles. I have indeed treated upon ſome parts of this ſubject in other books, and ſo have partly anticipated the ſubject. But I ſhall make abſtracts of theſe things, as I have occaſion, for the further proſecution of this matter. The practical part of muſic is well known, and in great perfection; but the ſpeculative part is but known to few, and is what I deſign to treat of in this paper.

DEFINITIONS.

Def. 1. *Muſic* is the ſcience of pleaſing ſounds, and conſiſts of melody and harmony.

Def. 2. *Melody* is the pleaſure ariſing from hearing a ſucceſſion of muſical ſounds, as in any tune or piece of muſic; this relates only to a ſingle part, and is the ſame as the *air* of a tune.

Def. 3. *Harmony* is the pleaſure ariſing upon hearing a compoſition of muſical ſounds in conjunction, or muſic in ſeveral parts.

Def. 4. A *note* or *tone*, theſe two words are uſed promiſcuouſly either to ſignify the *degree* of ſound, as it ſtands in the ſcale; or the *diſtance* between two of them.

Def. 5. The *ſcale* or *gamut* is a liſt or catalogue of all the notes in muſic, placed in their proper order.

Fig. Def. 6. *Pitch* is the height to which the ſcale is ſet; or the voice or any inſtrument.

Def. 7. *Key* or *key note*, is the loweſt or ending note of a piece of muſic. There are two of theſe, the *flat key* and the *ſharp key*.

Def. 8. *Cliffs* or *clefts* are marks ſet at the beginning of every piece of muſic, to denote what part it is, whether treble, baſe, &c. which ſaves the labour of writing it down.

Def. 9. *Moods*, certain marks uſed to denote the various meaſures of time.

Def. 10. *Cadences* are the concords and diſcords taken at the cloſe or concluſion of a piece of muſic, or of ſome part of it; by theſe the muſic naturally falls into, and terminates in the key.

Def. 11. *Cords* or *Concords*, are ſuch notes as produce a harmony, when they ſound together. Diſcords produce none.

Def. 12. *Mudulation*, is the art of paſſing out of one key into another.

Def. 13. *Tranſition*, is breaking a note into ſeveral leſſer notes, to ſweeten the roughneſs of a leap, by gradually paſſing from that note to the next.

PROP. I.

In any muſical ſtring, whoſe diameter $= d$, *length* $= l$, *weight* $= w$, *its tenſion or weight that ſtretches it* $= T$, *and* $f = 16.1$ *feet; then the time of its vibration will be* $\sqrt{\frac{wl}{2Tf}}$, *in ſeconds.*

This is demonſtrated in Prob. II. Sect. 3. Fluxions, to which I muſt refer the reader, for the proof of it.

Cor.

Cor. 1. *The number of vibrations performed in a second, is* $\sqrt{\frac{2Tf}{wl}}$. Fig.

Cor. 2. *The ſquare of the time of vibration of any muſical ſtring is as the length and weight directly, and tenſion reciprocally.*

Cor. 3. *If the matter is the ſame; the times of vibration of different muſical ſtrings, are as the diameters and lengths directly, and the ſquare root of the tenſions reciprocally.*

For w is as ddl, and therefore the time is as $\left(\sqrt{\frac{ddll}{T}} =\right) \frac{dl}{\sqrt{T}}$.

Cor. 4. *If the matter is the ſame, the number of vibrations in a ſecond, of different ſtrings, are as* $\frac{\sqrt{T}}{dl}$.

Cor. 5. *In the ſame ſtring and tenſion, the time of vibration will be as the length.*

Cor. 6. *In the ſame ſtring and tenſion, the number of vibrations in a ſecond, will be reciprocally as the length.*

Cor. 7. *If the matter be the ſame, and the tenſions as the ſquares of the diameters; the times of vibration will be as the lengths.*

Cor. 8. *If the matter and the length be the ſame; and the tenſions as the ſquares of the diameters; the times of vibrations will be equal; and they will give the ſame ſound.*

SCHOL.

By this laſt Corol. the ſtrings ought to bear the ſame pitch, and both break together. But I always found the thickeſt to break firſt, and the ſmalleſt to bear the higheſt pitch.

It

Fig. It may not be amiſs here to relate ſome experiments I made of the vibrations of ſtrings.

I took a wire ſtring 57.95 inches long, weighing $87\frac{2}{4}$ grains, which faſtened to a virginal, and ſtrained with 7*lb.* weight; a quarter of it was uniſon with EE flat in the baſe. By this theory it made 65 vibrations in a ſecond; and therefore 4×65 or 260 is the number of vibrations which the ſtring E flat (in the mean) makes. But this inſtrument was a note below pitch.

A ſtring 52.9 inches long, weight 71 grains, ſtrained with $10\frac{1}{2}$*lb.* on a ſpinet, half of it was uniſon with AA in the baſe, whence the whole made 86.7 vibrations in a ſecond. And 4×86.7 or 347 is the number of vibrations which A (high in the baſe) makes in a ſecond, and E flat makes 245.

At 7*lb.* weight it was with F, which makes 280 vibrations, whence A would make 353, and E flat 250.

At 9*lb.* weight it was with G, which makes 322 vibrations, whence A would make 362, and E flat 255.

By a former experiment E flat was 254. Therefore at a mean of 5 experiments, E flat made 253 vibrations, and C below made 212.7.

PROP. II.

If two muſical ſtrings be made to vibrate and ſound together; the ſweetneſs of the harmony will be greater or leſs, according as the coincidence of their vibrations are many or few. And if their vibrations never coincide, they produce the moſt harſh and diſagreeable diſcord.

69. Theſe things are matter of experience. For let AB, *ab* be two ſtrings of the ſame matter and bigneſs for length and breadth, and ſtretched with the

ſame

ſame weight, and made to vibrate; their vibrations will be compleated in the ſame time, by Prop. I. and they will both give the ſame ſound, whether they ſound together or ſeparate. Here their vibrations always coincide, and their united ſound is *uniſon*, which is the moſt perfect harmony that can be. Fig. 59.

Now let one of the ſtrings *ab* be ſtopt in the middle at *c*, and AB and *cb* be made to vibrate; then (Cor. 4. Prop. I.) *cb* will vibrate juſt twice as faſt as AB; and their vibrations will coincide at every vibration of AB, and at every other of *cb*; and this is the ſweeteſt harmony that can be made by ſtrings of different lengths, and is called an *octave* or *eight*. And there is ſo much likeneſs in the idea or perception of theſe two ſounds, that muſicians do not ſcruple to account all eights the ſame. In the ſame manner if *cb* was ſtopt in the middle, and made to vibrate with AB, it would vibrate four times as faſt, and every 4th of its vibrations would coincide with every vibration of AB; and the ſame harmony, and the ſame ſimilitude, will ſtill remain.

Again, let *ad* be a third part of *ab*; and the ſtring being ſtopt at *d*, let AB and *db* vibrate together, then every third vibration of *db* will coincide with every ſecond of AB; and produce a delightful harmony, which is the next agreeable concord, and is called a *fifth*.

In like manner if *af* be a fourth part of *ab*; and AB, *fb* vibrate together; their vibrations will coincide at every fourth vibration of *fb*, and every third of AB; which produces that agreeable harmony called a *fourth*.

After the ſame manner, if *ag* be a fifth part, and *ah* a ſixth part of *ab*; and *gb* or *hb* be made to vibrate with AB; then the fifth vibration of *gb* will always coincide with the fourth of AB; and the

Fig. the ſixth vibration of *bb*, with the fifth of AB; and they will ſtill produce a delightful harmony, but leſs and leſs perfect. Theſe are called the *greater* and *leſſer thirds*.

But let a part of *ab* be ſtopt, which is incommenſurable to AB, the ſound of theſe, vibrating together, will produce nothing but a diſagreeable noiſe, without the leaſt harmony. So it is evident that it is the coincidence of the vibrations that cauſes the harmony; and the more frequent that coincidence is, the greater is the harmony; and where there is no coincidence, there is no harmony at all.

70. Cor. 1. *If there be four ſtrings of equal ſize and tenſion, whoſe lengths are ba, bg, bd, bc; and all made to ſound together, they will produce the moſt agreeable harmony, that is poſſible to be made with four ſtrings, in the compaſs of an octave.*

For if the length *ba* be 1; *bg* will be $\frac{4}{5}$; *bd*, $\frac{2}{3}$; and *bc*, $\frac{1}{2}$. Therefore 4 vibrations of *ba* are equal to 5 of *bg*, and to 6 of *bd*; and to 8 of *bc*.

Cor. 2. *And the greateſt harmony that can be made with 6 ſuch ſtrings, will be when their lengths are* 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, *and* $\frac{1}{6}$.

For the number of vibrations will be reciprocally as their lengths; whence 1 vibration of the firſt, will be equal 2, 3, 4, 5, 6, vibrations of the 2^d, 3^d, 4^{th}, 5^{th}, 6^{th}, reſpectively. And after theſe periods they all conſtantly coincide.

Cor. 3. *If ſeveral parts of the ſtring ab (which denote any cords) be multiplied together; the product will denote the cord, which is the ſum of all the reſt. Or they may be diſcords.*

For if $gb = \frac{8}{9}$, and $db = \frac{3}{4}$, then $\frac{8}{9} \times \frac{3}{4} = \frac{2}{3}$, that is, a whole tone and a fourth, make a fifth. For the ſtring *gb*, ſtopt at *d*, produces a fourth above

above *g*, or a fifth above *a*. In like manner $\frac{2}{6} \times \frac{4}{5}$ Fig. $\times \frac{3}{4} = \frac{1}{2}$, that is a leſſer third, a greater third, and a fourth, make an eight. And ſo of others.

PROP. III.

If ZA *be any muſical ſtring; and if* ZA, Z*b*, Z*c*, Z*d*, Z*e*, *&c. be taken in geometrical progreſſion; and if* ZA *be made to vibrate, and then ſtopt ſucceſſively at b, c, d, e, f, &c. The ſounds produced thereby will aſcend equally, or riſe by equal degrees.* 71.

That the ſound of any muſical ſtring is cauſed by the vibrating motion thereof ſtriking the air, is matter of experience; and likewiſe the more any ſtring is ſtretched, the faſter it vibrates, and the acuter the ſound will be. And when the ſtretching force or tenſion is the ſame; the ſhorter the ſtring is, the more acute the note will be, and the vibrations quicker.

It is alſo known by experiments, that in any ſtring whatever, if it be ſtopt in the middle, it will then ſound an eighth higher. If a third part be ſtopt, it will ſound a fifth higher. If a fourth part be ſtopt, it will ſound a fourth higher. If a fifth part be ſtopt, it will ſound higher by a greater third. If a ſixth part be ſtopt, it will ſound higher by a leſſer third; and ſo on, whatever be the length of the ſtring: for the acuteneſs of the tone depends upon the quickneſs of the vibrations. Therefore if AZ, *b*Z, be two ſtrings, or one ſtring ſtopt at *b*, and if A*b* be ſome determined part of AB (as $\frac{1}{6}$); then the ſound of *b*Z is raiſed a determined degree (a leſſer third): for the ſame reaſon if *b*Z be a ſtring, and *bc* the ſame determined part of *b*Z; then ſtopping it at *c*, will alſo raiſe the ſound by the ſame degree. And if *c*Z be a ſtring, and *cd* ſtill the ſame determined part of *c*Z; ſtopping

Fig. 71. ping at *d* will alſo raiſe the ſound an equal degree higher; and ſo on. Now A*b* is the ſame part of AZ, as *bc* is of *b*Z, and as *cd* is of *c*Z, &c. becauſe ZA, Z*b*, Z*c*, Z*d*, &c. are in geometrical progreſſion, and A*b*, *bc*, *cd*, &c. in the ſame geometrical progreſſion. Therefore the ſtring AZ, ſtopt at *b*, *c*, *d*, *e*, &c. will raiſe the ſound gradually by equal degrees.

Cor. 1. *Hence* $\frac{Ab}{AZ} = \frac{bc}{bZ} = \frac{cd}{cZ}$ *&c. and each of them may denote that given degree, the ſound is raiſed, or elevated in ſtopping at b, c, d, &c.*

Cor. 2. *And conſequently, when any part as* A*d is but ſmall, the height the ſound is raiſed by the ſtrings* AZ, *d*Z, *is nearly as* $\frac{Ad}{AZ}$.

For then A*b*, *bc*, *cd*, will be nearly equal. But that degree of aſcent will be more exactly $\frac{Ad}{\frac{1}{2}AZ + dZ}$.

Cor. 3. *If* A*b*, *bc*, *cd*, *de*, *&c. be taken extremely ſmall; any part of the ſtring as* A*g will be analogous to the logarithmic area; and the riſe of ſound, to the logarithm correſponding.*

For A*b*, *bc*, *cd*, &c. being in geometrical progreſſion, and the correſpondent aſcents of ſound all equal, or the whole in arithmetic progreſſion. Theſe laſt will be the logarithms, and the former will fill up the logarithmic ſpace.

Cor 4. *Let* AZ = 1 *the length of the whole ſtring.*
Z*g* = *l*, *any other ſtring.*
e = *elevation of its tone (above* ZA.)
E = *elevation of tone in an octave.*

Then $e = \frac{E \times log.\ l}{.30103}$ *the elevation of tone when ſtopt at g.*

For

For ZA, Z*b*, Z*c*, Z*d*, &c. being geometrical proportionals; and the correſpondent elevations of ſound, arithmetical proportionals; the former will be the numbers, and the latter their logarithms. And ſince the ſtring $\frac{1}{2}$ is an octave to AZ, and E its elevation of tone; we ſhall have log. $\frac{1}{2}$ (.30103) : E : : log. : Z*g* (log. *l*) : *e*. Fig.

Cor. 5. *If two ſtrings be in a given ratio, or if one be ſtopt in a given ratio; the ſound will be raiſed a given height.*

For a ſtring double in length to another, or ſtopt in the middle, the ſound will riſe an eighth. If $\frac{1}{3}$ be ſtopt, it riſes a fifth. If $\frac{1}{4}$ be ſtopt, it riſes a fourth; if $\frac{1}{5}$ be ſtopt, it riſes a third, &c.

PROP. IV.

To inveſtigate the number and diſtances of all the notes in an octave, and therefore in the whole ſcale. 70.

This is to be done by Prop. II. and its Corollaries, but moſt eaſily by the firſt Corol. By this Cor. the four ſtrings whoſe lengths are 1, $\frac{4}{5}$, $\frac{2}{3}$, and $\frac{1}{2}$ ſounding together, give the moſt agreeable harmony. And if they be ſo delightful when taken in conjunction, they cannot be otherwiſe when taken in ſucceſſion; therefore we may be aſſured, that theſe four notes actually exiſt in the ſcale. Then our next buſineſs is to find at what diſtances the two ſounds in the middle (given by the ſtrings $\frac{4}{5}$, $\frac{2}{3}$) are placed above the ſound of the loweſt (given by 1) which is the ground; for that given by the ſtring $\frac{1}{2}$ is fixt, being an octave higher. The four ſtrings ſounding theſe notes are *ba*, *bg*, *bd*, *bc*; then calculating the diſtances of ſound by Cor. 4. of the laſt Prop. we ſhall find that the ſtring *ba*,

ſtopt

Fig. 70. ſtopt at *g*, raiſes the ſound .322 parts of an octave above *a*.

ſtopt at *d*, raiſes the ſound .263 parts of an octave above *g*.

ſtopt at *c*, raiſes the ſound .415 parts of an octave above *d*;

where *d* is an octave to the firſt at *a*. But theſe 3 diviſions (if they were all equal) would only make 3 notes in an octave, which would be too few to expreſs any muſic, as there could be no variety therein. Therefore we muſt ſubdivide theſe diviſions into leſs ones, as near equal as we can, ſo as to be melodious; for that is the ultimate end of all our enquiries. If the leaſt $gd = 263$ be taken for the length of a note, *ag* will contain about $1\frac{1}{4}$, which does not anſwer right. If *gd* contains 2 notes, then *dc* will contain $3\frac{1}{2}$, which ſtill does not anſwer. If *ag* contains 4, then *dc* will contain about 5, and *gd* about 3. This is the neareſt diviſion we can get in ſmall numbers. According to which there will be 12 diviſions, or 12 notes in the octave almoſt equal to one another; only theſe in *ag* will be ſomething leſs than the reſt. But here will be too many notes here in the compaſs of an octave; for if 12 ſuch ſtrings be made to ſound in ſucceſſion, they will give no delight at all; therefore this is not the thing that nature aims at. Therefore we muſt try to reduce them to a leſſer number; and this is not to be done otherwiſe than by taking half of *ag* for a note; then

we have in *ag* 2
in *gd* $1\frac{1}{2}$
in *dc* $2\frac{1}{2}$

There will then be in all 5 whole ſpaces and two half ones; but theſe half ſpaces cannot be both put into one, to make a whole ſpace, becauſe they happen in different diviſions, one being in *gd*, the other in *dc*, which [illegible] with one another.

other. Here then we ſhall have the octave divid- Fig.
ed into 7 parts, 5 of them greater, and nearly 70.
equal to one another; and 2 leſſer alſo nearly equal, and about half the bigneſs of the former. The greater may be properly called *whole notes*; and the leſſer *half notes*. And nature will admit of no more, without the utmoſt jangling of the ſounds, which cannot be called muſic.

The next thing is to find the places of theſe half notes, ſo that one be in the diviſion *gd*, and the other in the diviſion *dc*. And ſince this is entirely arbitrary, we muſt call in nature to our aſſiſtance. Let them then be combined together all the ways poſſible by the laws of *Permutation*, ſo that the half notes keep their proper diviſions. And here they can only be combined 6 different ways. Then let each of them be tried, by ſinging or playing them gradually down from the higheſt to the loweſt, in the nature of ringing 8 bells; and obſerve which muſic is the moſt melodious and delightful, and take that combination for the true order of the notes in an octave; and that will be found to be $\frac{1}{2}$, 1, 1, 1, $\frac{1}{2}$, 1, 1; no other combination being ſo muſical. So that here nature and art have jointly conſpired to find out the true ſyſtem of notes contained in the gamut, on which all muſic is built. If the notes and half notes be continued through ſeveral octaves, the notes will ſtand thus,

1, 1, $\frac{1}{2}$, 1, 1, 1, $\frac{1}{2}$, 1, 1, $\frac{1}{2}$, 1, 1, 1, $\frac{1}{2}$, 1, 1, $\frac{1}{2}$, 1, 1, 1, $\frac{1}{2}$, &c.

where the two half notes are ſo placed, *that there are always two and three whole notes alternately between them. And this is an eſſential property of the ſcale or gamut.* And I affirm, that all tunes, or pieces of muſic whatſoever, muſt be played by a ſcale thus conſtructed; and that nature allows of no other ſo perfect and melodious.

Indeed it is almoſt ſelf-evident, that to have the ſeries of notes as melodious as poſſible, that the

Fig. 70. half notes be placed as far off one another as they poſſibly can; ſo that there muſt be two and three whole notes alternately placed between every two half notes. For any body may find, by trying it upon an inſtrument, how diſagreeable the muſic is when the half notes are placed together, or only parted by one note.

Further, theſe half notes muſt keep their places, in reſpect of the loweſt note, which note therefore is the *key*. For let as many changes as you will be rung on 8 bells; the muſic naturally tends to end upon the loweſt note; and it is the ſame thing for any tune. That note then is the key, that has two whole notes above it, and then a half note; and a half note below it.

The half notes give a charming and pleaſing variety in all muſic. If all the notes had been by nature equal, there would not have been that variety and ſweetneſs, which the mixture of whole notes and half notes affords.

Now for diſtinction, names muſt be given to theſe 7 notes; and they are denoted by the ſeven letters of the alphabet; and that which is the key is called C, and the reſt in order; which being placed in proper order, and at their due diſtances to ſhew the half notes; the gamut will be as follows,

C D EF G A BC D EF G A BC D EF &c.

So every eight has the ſame letters repeated; how the notes, that they repreſent, are to be written for the practice of playing, will be ſhewn afterwards. If the notes in an octave be reduced to numbers, they will ſtand thus,

$$\overset{\text{E}}{\text{C}}\ \frac{8}{9}\times\frac{9}{10}\times\frac{15}{16}\times\frac{8}{9}\times\frac{9}{10}\times\frac{8}{9}\times\frac{15}{16}\ \overset{\text{G}}{c}$$

But $\frac{8}{9}$ and $\frac{9}{10}$ may change places, in the period or diviſion CE; and alſo in the diviſion G*c*. Theſe fractional numbers denote the parts of the ſtrings

to

to ſound the notes in progreſſion, each part taken from the laſt length. That is (if ZA be the ſtring, Fig. 72.) $\frac{8}{9}$ of AZ, $\frac{9}{10}$ of *b*Z, $\frac{15}{16}$ of *c*Z, &c. are the parts. Or the whole lengths ſounding the ſeveral notes will be 1, $\frac{8}{9}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{8}{15}$, $\frac{1}{2}$; that is $ZA = 1$, $Zb = \frac{8}{9}$, $Zc = \frac{4}{5}$, $Zd = \frac{3}{4}$, &c. of ZA, the whole ſtring. And hence $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{10}$, $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{9}$, $\frac{1}{10}$, repreſent the length of the notes, from the key, being equal to A*b*, *bc*, *cd*, *de*, &c. Fig. 72.

Cor. 1. *It is an eſſential property of the ſcale of muſic, that through the whole, there are placed alternately two and three whole notes, between every two half notes.*

Cor. 2. *It follows from what has been ſaid, that* C *is the true key of all muſical compoſitions that are gay, lively and merry; or that ſuch pieces of muſic will all naturally end in that note, and is called the ſharp key.*

SCHOL.

It may be obſerved, that another ſeries of notes might have been taken to make up an octave; and that is, inſtead of half notes, to join them to the next whole notes; and then we ſhould have 5 notes in the octave; 3 leſſer nearly equal, and two greater, being half as much more as the other; and this ſcale may be ſaid to be the former ſcale abridged or contracted. And they will ſtand as follows,

C D E G A C *d e g* &c.

in numbers $\frac{8}{9}$ $\frac{9}{10}$ $\frac{5}{6}$ $\frac{9}{10}$ $\frac{5}{6}$

Here one and two of the proper notes ſtand alternately between each two of the double notes. It cannot be denied, but that muſic may be compoſed from this ſyſtem; but by reaſon of too few notes in the octave, the muſic will not have compaſs and variety enough; nor will it have that ſweetneſs which it would have by the ſyſtem of 7 notes; as

Fig. 72. any one may experience upon an inſtrument, by ſtriking theſe 5 notes like ringing 5 bells.

What has been ſaid here, relates to the gay key which ends in C; but there is another key in which all grave and ſolemn pieces of muſic always end, which is A, as will be ſhewn in the next Prop. But the ſyſtem of notes is the ſame for both keys.

Notwithſtanding the imperfection of this ſcale, yet ſix bells made to ſtrike theſe notes, would go very prettily.

PROP. V.

As the note C *in the gamut, is the key in which all merry and joyful muſic ends; ſo the note* A *is the key in which all flat and ſorrowful muſic ends, and to which it tends as to a center.*

This is proved by experiments, for the flat key, in the ſame manner as was done for the ſharp key. For let any tune or piece or muſic, which excites grief or pity be tried, and it will be found always to tend to A at its concluſion, and to end there. And as this is found always to hold good; then this is naturally the key to all theſe ſoft movements. I confeſs I can give no reaſon for this, no more than I can give a reaſon why ſugar is ſweet, and wormwood bitter; experience ſhews both. But I think it is no wonder at all to find that theſe pieces of muſic which are ſprightly, and thoſe that are doleful ſhould tend to different points, and have different concluſions. When a perſon in diſtreſs is complaining, or lamenting in tears, the voice naturally riſes from B to C, and then drops into A. This is the muſic of a mournful languiſhing voice, as in the crying of a child. And as grief by ſome mechaniſm in nature, excites this ſort of vocal muſic; ſo on the contrary, this kind of

of muſic excites grief, ſorrow, melancholy, flatneſs, and uneaſineſs in the mind. Fig.

Cor. 1. *Muſic in the ſharp key, or in* C, *is gay, lively, chearful, ſprightly, merry; and inſpires joy and gladneſs. It is known by having a greater third above it; or only two whole notes above, and a half note below.*

Cor. 2. *Muſic in the flat key* A *is ſoft, grave, ſolemn, dull, penſive, ſorrowful, melancholy, doleful; and excites grief, pity, woe, ſadneſs, mourning, &c. It is known by having a leſſer third above it; or a whole note above it, and* 2 *whole notes below.*

SCHOL.

Theſe two keys are ſo different in their nature, that one inſpires mirth and the other ſadneſs; and that in different degrees. There are certain ſounds natural to joy, others to ſorrow and deſpair; others to love and compaſſion, &c. And a ſkilful artiſt in muſic can make ſuch compoſitions, or combinations of notes, as can raiſe any of the paſſions of the human ſoul. Various combinations of ſounds have various effects upon the imagination. Some compoſitions inſpire courage, others terror. Some incline to love, joy, and mirth; others to anger, ſorrow, deſpair, &c. A judicious mixture of concords and diſcords have ſo bewitching and magical effects upon the paſſions, and ſo powerfully excite them, and work upon the imagination ſo ſtrongly, that men frequently run into actions of a correſpondent nature, or ſuch as are the natural reſult of courage, anger, love, deſpair, &c. So ſurprizingly do theſe artful compoſitions in muſic raiſe or depreſs the ſpirits.

Fig.

PROP. VI.

To determine the number of concords and diſcords.

It has been obſerved (Prop. II.) that a concord is cauſed by the frequent coincidence of the vibrations of two ſtrings; and a diſcord by their ſeldom or never coinciding. This being allowed, we muſt conſider how many caſes can happen where the lengths of two ſtrings can be expreſſed by ſmall numbers; for their ratio denotes ſome concord.

1. The length of two ſtrings being 1 and 2, $\frac{1}{2}$ denotes an eighth.

2. The lengths being 2 and 3, then $\frac{2}{3}$ expreſſes a fifth.

3. The lengths being 3 and 4, then $\frac{3}{4}$ denotes a fourth.

4. If the lengths be 4 and 5, then $\frac{4}{5}$ is the greater third.

All theſe have been ſhewn in Prop. II.

5. If the lengths be 3 and 5, then $\frac{3}{5}$ is the concord, and (by Cor. 3. Prop. II.) that is equal to $\frac{3}{4} \times \frac{4}{5}$, conſequently $\frac{3}{5}$ is a greater ſixth, being equal to a fourth, and a greater third; $\frac{2}{5}$ is too little, being leſs than $\frac{1}{2}$, which denotes an eighth.

6. If the lengths be 5 and 6, then $\frac{5}{6}$ denotes a leſſer third (Prop. II.) If the lengths be 4 and 6, then $\frac{4}{6} = \frac{2}{3}$, which we had before. And $\frac{3}{6}$ is $\frac{1}{2}$ as before.

7. If the lengths are 6 and 7, or 5 and 7, or 4 and 7; I doubt not but theſe would make imperfect concords. But they can be of no uſe in our ſcale, as they would be incommenſurable to the reſt of the notes.

8. If the lengths be 7 and 8, this concord would alſo be incommenſurable and uſeleſs. And $\frac{6}{8}$ is equal to $\frac{3}{4}$, which we had before. But if the lengths

lengths be 5 and 8, then $\frac{5}{8}$ will be a concord, being $= \frac{5}{6} \times \frac{6}{8} = \frac{5}{6} \times \frac{3}{4}$, and is equal to a leſſer third and a fourth, and therefore is a leſſer ſixth. Fig.

9. If we take 8 and 9, we come among the diſcords; for $\frac{8}{9}$ is but the diſtance of a note, and $\frac{9}{10}$ another note, alſo a diſcord; and all beyond are diſcords; till you come at $\frac{15}{16}$, which is ſtill a greater diſcord. Yet notwithſtanding theſe go under the name of diſcords, they are not properly ſuch; for they produce a ſort of harmony, for in the ſtrings 8 and 9, the vibrations coincide at every 8th of one, and every 9th of the other; and that coincidence is the criterion of the degree of harmony. But theſe that have no ſuch coincidence are properly diſcords. And hence it comes that theſe diſcords $\frac{8}{9}$, $\frac{9}{10}$, &c. mixt with concords, make the moſt agreeable muſic.

The number of concords will now appear to be,

1. The *leſſer third* $(\frac{5}{6})$, conſiſting of a note and a half.

2. The *greater third* $(\frac{4}{5})$, containing two notes.

3. The *fourth* $(\frac{3}{4})$, contains two notes and a half.

4. The *fifth* $(\frac{2}{3})$, conſiſts of 3 notes and a half.

5. The *leſſer ſixth* $(\frac{5}{8})$, contains 4 notes, or 3 notes and two half notes.

6. The *greater ſixth* $(\frac{3}{5})$ conſiſts of 4 notes and a half.

7. The *eighth* $(\frac{1}{2})$, conſiſts of 6 notes, or 5 notes and 2 half notes.

The number of diſcords are, the *leſſer ſecond* $\left(\frac{15}{16}\right)$, the *greater ſecond* $\left(\frac{8}{9} \text{ or } \frac{9}{10}\right)$, the *leſſer ſeventh* $\left(\frac{10}{18} = \frac{5}{9}\right)$, the *greater ſeventh* $\left(\frac{16}{30} = \frac{8}{15}\right)$. The greater fourth is a diſcord when it conſiſts of 3 notes, and is the ſame as a flat fifth, as from B to F. So that the number of concords is 7; and

Fig. the number of diſcords 5. If we go without the compaſs of an eighth, it comes to the ſame thing; for an eighth, ninth, tenth, &c. are the ſame as the uniſon, ſecond, third, &c. The 8th and 5th are called perfect concords; the reſt imperfect.

Cor. 1. *The leſſer ſixth is equal to a leſſer third and a fourth; a greater ſixth is equal to a greater third and a fourth. An eighth is equal to a fifth and a fourth, and to a leſſer third and greater ſixth, and to a greater third and leſſer ſixth, and to a ſecond and ſeventh; which may be called complements to one another.*

Cor. 2. *Of concords, the 8th is moſt perfect, then the 5th, and next the thirds, and then the ſixths.*

Cor. 3. *Of diſcords, the leſſer ſeventh and greater ſecond are the ſweeteſt, being complements. And the greater ſeventh and leſs ſecond the moſt harſh, being alſo complements. The fourth (greater) is alſo harſh.*

PROP. VII.

To find the diviſions of a monocord to ſound the concords, according to the number of notes contained in each; upon ſuppoſition, that all the notes in the gamut riſe by equal degrees; and to ſhew the difference between theſe and the harmonic diviſions; or between the iſotonic and harmonic ſcale.

This Problem is ſolved at large in Prob. XXI. *Algebra*, and in Ex. 108. of the *Mechan.* 4to. The particulars of which I ſhall here put down. There are 6 notes contained in an octave, or 12 half notes. But theſe notes are not all of a bigneſs in all the concords, as is ſhewn in the Mechanics; ſo the notes do not riſe therein by equal intervals of ſound, according to that harmonic diviſion. Let

ZA be the ſtring $= 1$, $Zb = \frac{1}{2}$. And let Zb, Zc, Zd, &c. found the half notes, and put $Zb = x$, then Zb is the firſt of 12 geometrical proportionals; whence is found $x = \sqrt[12]{\frac{1}{2}} = .9439$ for half a note; and $Zc = xx = .8909$ for a note. And $x^3 = 8409 = 1\frac{1}{2}$ notes, $x^4 = .7937 = 2$ notes, $x^5 = .74915 = 2\frac{1}{2}$ notes; $x^7 = .66742 = 3\frac{1}{2}$ notes, $x^8 = .62996 = 4$ notes, $x^9 = .5946 = 4\frac{1}{2}$ notes, $x^{12} = .5 = 6$ notes. Theſe are the diviſions for each of the concords in order, for aſcending equally. Fig. 71.

But the harmonic diviſions will be $.8\frac{1}{2}$, .8, .75, $\frac{2}{3}$, .625, .6, $\frac{1}{2}$. Theſe put into decimals, and ſet down in a Table with the former, will be as follows.

names of the cords.	pure concords.	equal aſcents.	errors.
wh. ſtring.	1.00000	1.00000	0
leſſer third	.83333	.84090	$\flat \frac{1}{19}$
greater third	.80000	.79370	$\sharp \frac{1}{13}$
fourth	.75000	.74915	$\sharp \frac{1}{100}$
fifth	.66666	.66742	$\flat \frac{1}{100}$
leſſer ſixth	.62500	.62996	$\flat \frac{1}{15}$
greater ſixth	.60000	.59460	$\sharp \frac{1}{13}$
eighth	.50000	.50000	0

Then to find the errors or variations of the correſpondent cords. Among the methods made uſe of in the Algebra, we have this (which is alſo in the Mechanics); Ac or the length of a note at A is $= 1 - xx = .1091$; and the length of a note at any other place as d, that is $de = Zd \times .1091 = P$. And let Zd repreſent any of the numbers in the third column, and s that in the ſecond, correſponding. Then the error will be $\frac{Zd - s}{P}$, for that concord. As in the fourth, $Zd = .74915$, $s =$

Fig. 71. $s = .75000$, and $P = .74915 \times .1091 = .0817$; and $Zd - s = .0008$, then $\frac{Zd-s}{P} = \frac{0008}{.0817} = \frac{1}{100}$ nearly, the error in the fourth. The errors in all the concords being thus computed, are ſet down in the Table, col. 4th. This laſt column ſhews the quantity of the error, or what part of a note it is; and the mark ♯ ſhews the note (by equal aſcent) to be too ſharp, and the mark ♭ ſhews when it is too flat. Here theſe concords that are the complements to one another have the ſame error, but of different denominations.

Cor. 1. *Hence, in this iſotonic ſcale, ſince an eighth contains 6 whole tones; then computing from the number of notes in each concord, as found in the laſt Prop. we ſhall find 4 leſſer thirds = an eighth, 3 greater thirds = an eighth, 12 fourths = 5 octaves, 12 fifths = 7 octaves, 3 leſſer ſixths = 2 octaves, and 4 greater ſixths = 3 octaves.*

Cor. 2. *The iſotonic ſcale, as here deſcribed, is the beſt ſcale for practice, that can be contrived.*

For in the ſcale deſcribed in Prop. IV. which is the natural ſcale, and the only one that nature has laid down; the notes are not all equal, ſome are denoted by $\frac{8}{9}$ and ſome by $\frac{9}{10}$, the difference is $\frac{1}{90}$, or a tenth part of one of them; which makes no ſenſible difference in playing upon a ſingle inſtrument. But in compoſition, it does not anſwer; for if a concord can be taken perfect from ſome of the notes, it will be imperfect when taken from others, which occaſions great confuſion; but the iſotonic ſcale anſwers equally for all; and the difference from the true ſcale is inſenſible. For the greateſt error is in the leſſer third and greater ſixth, which amounts but to the thirteenth part of a note, which makes an imperceptible variation from the true

cord,

cord, when played in conſort; and which no ear can diſcover. For ſuch an error is not ſo diſcoverable in a third or ſixth, as in a fifth or eighth; for the ſweeter the cord, the ſooner an imperfection is diſcovered. But the error in a fifth muſt be quite inſenſible, being but the hundred part of a note. And although theſe trifling errors will a little diminiſh the ſweetneſs of the harmony, yet there is no remedy, but what is worſe than the diſeaſe. Fig.

In the natural ſcale, the difference of a fourth and fifth is $\left(\frac{3}{4} - \frac{2}{3}\right)\frac{1}{12}$, for the ſtring $\frac{3}{4}$; or $\frac{1}{9}$ for the ſtring 1; therefore $\frac{8}{9}$ will be a note, and therefore $\frac{\overline{8}|^{6}}{9}$ ſhould be $= \frac{1}{2}$, but it is only .4933. For there are 6 notes in the octave.

Again, take the difference between the fourth and leſſer third, and we ſhall have $\frac{9}{10}$ for a tone. But $\frac{\overline{9}|^{6}}{10} = .5314$ inſtead of $\frac{1}{2}$. So that neither $\frac{8}{9}$ nor $\frac{9}{10}$ is an exact tone.

A leſſer third, a greater third, and a fourth, differ by half a tone, and there ſhould be 12 in the octave. In the firſt we have $\left(\frac{5}{6} - \frac{4}{5}\right)\frac{1}{30}$ for the ſtring $\frac{5}{6}$ or $\frac{1}{25}$ for the ſtring 1, and the ſtring half a note above is $\frac{24}{25}$, but $\frac{\overline{24}|^{12}}{25} = .6127$, which ſhould be $\frac{1}{2}$. And in the ſecond $\left(\frac{4}{5} - \frac{3}{4}\right)\frac{1}{20}$ for the ſtring $\frac{4}{5}$, or $\frac{1}{15}$ for the ſtring 1. And $\frac{\overline{15}|^{12}}{16} = .4609$, which ſhould be $\frac{1}{2}$. Therefore the firſt is too little, and the laſt too big, for a half note.

As none of theſe notes or half notes will make up an octave, ſo neither will any number of thirds, fourths,

Fig. fourths, or fifths, make one or more octaves. For then (by Cor. 1.) we ſhould have $\frac{5}{6}^{\overline{|4}}$, or $\frac{4}{5}^{\overline{|3}}$, or $\frac{3}{4}^{\overline{|\frac{12}{5}}}$, or $\frac{2}{3}^{\overline{|\frac{24}{7}}} = \frac{2}{1}$; but none of theſe are ſo; and therefore we may conclude, that no ſcale made up of ſuch notes or half notes, or any combinations of them, or of perfect thirds, fourths, fifths, &c. can be perfectly exact. And therefore the iſotonic ſcale, whoſe numbers are in the third column of the Table, muſt be the beſt and moſt practical ſyſtem.

They that from the diviſion of the monocord inſiſt, that all inſtruments ſhould be turned according to that diviſion, which is $\frac{8}{9} \cdot \frac{9}{10} \cdot \frac{15}{16} \cdot \frac{8}{9} \cdot \frac{9}{10} \cdot \frac{8}{9} \cdot \frac{15}{16}$, don't conſider, that thereby they bring the practice of muſic into a very narrow compaſs. For it muſt always be performed only upon one key. For the notes can never have the ſame poſition, but in that ſituation of the key.

Cor. 3. *Therefore all inſtruments ought to be tuned according to the iſotonic ſcale. And in the harpſichord, &c. when tuned by fifths, the upper note muſt be taken as flat as the ear will bear, and alſo leſſer thirds; but greater thirds muſt be taken as ſharp as they will bear.*

It is the practice of moſt people in tuning, to take the fifths perfect, till the laſt, which they leave imperfect; and ſuch imperfect notes they call *bearing notes*. But this ſpoils the muſic, for a fifth taken on that note is no better than a diſcord, and therefore ought to be avoided in compoſition, which will cramp the muſic, and perhaps quite ſpoil the air of the piece. For there will be 12 removes

or

or repetitions before one comes to the ſame note again; and ſince in one fifth there is $\frac{1}{100}$ error, in 12 fifths there will be $\frac{12}{100}$ or $\frac{1}{8}$ of a note at the laſt, which will be very ſenſible in a fifth; and the errors will be far greater in a third or a ſixth. Therefore thoſe that chuſe to have ſome of the concords perfect, muſt be forced to make others as imperfect, and often ſo bad, as to be no better than diſcords. Fig.

Another diſadvantage would be, that the muſic could not be tranſpoſed upon any other key, as occaſion required; but tied down to a very few; and upon theſe the ſame inconveniencies would happen.

Some people, to cure the imperfection of the ſcale, have invented quarter notes, which makes the muſic hard to play, and does not take away the diſeaſe neither. Therefore all things conſidered, I am perſuaded that the iſotonic ſcale is the beſt for practice, as it is attended with no inconveniencies at all. The errors ariſing by uſing that ſcale, does not amount to ſo much as the alteration of the weather, by heat or cold, drought or moiſture, cauſes in a little time, either upon ſtrings or pipes. Therefore the beſt way to remedy theſe errors, is to divide the defect equally among the notes of the inſtrument; ſo that what might be ſenſible in one note, may eſcape the ſenſes when divided among many.

The proper way to tune then is this; begin at the middle of the inſtrument, and firſt tune the fifth perfect, then lower the upper note a ſmall matter; do the ſame with all the reſt till the octave be finiſhed. Then if the laſt note be either too high or too low, begin anew, and accordingly alter all the notes a little, till the laſt does agree; but this art is gained by practice. This octave ought to be tuned exactly, for the reſt depend upon that; and there

Fig. notes remain as before. But then the reſt of the half notes are not to be played, and are in that caſe uſeleſs; for in every caſe there are only 7 notes to be played in every octave, the reſt having nothing to do in the muſic. For ſuppoſe there is a tune that ends in C, or has its key in C; then all the flats and ſharps, you ſee there are uſeleſs, and muſt be omitted. But ſuppoſe that tune to have its key in D, then you muſt have C♯ the next note below, and C muſt be left out. Likewiſe you have F♯, and F muſt be left out; and moreover all the other flats and ſharps that you ſee, muſt be left out, for they have no buſineſs. And thus the proper ſituation of the two half notes is preſerved the ſame as before, when the key was in C.

In modulation, or paſſing from one key to another, nothing can be done without proper flats or ſharps. For by the judicious uſe of ſome of theſe, the muſician paſſes almoſt imperceptibly from one key to another; and then he uſes ſuch flats or ſharps as are proper for that key, and all the reſt muſt be let alone; and alſo the proper notes belonging to ſuch flats and ſharps as he uſes, muſt be omitted. This modulation, or paſſing from one key to another in playing, is the ſame thing as tranſpoſition, or removing a tune from one key to another in writing or pricking. The rules whereof I ſhall ſhew in the next Prop.

PROP. IX.

To ſhew the nature, reaſon, and practice of tranſpoſition.

Tranſpoſition is the writing any piece of muſic higher or lower upon the ſcale, ſo as to anſwer the ſame purpoſe in playing. The art of tranſpoſing conſiſts

conſiſts in ſetting ſuch a number of flats or ſharps as ſhall keep the two half notes at ſuch a height and diſtance above the new key, as they had in the former. And the flats or ſharps muſt be ſo placed, and no otherwiſe, that there may be always two and three whole notes alternately between every two half notes, which is to be done after the manner I am now going to ſhew. The notes in the natural ſcale ſtand thus (*n.* 1); and where the half notes are, the letters are ſet cloſe together. Fig.

1. A BC D EF G A BC D EF G &c.
2. A BC D E FG A BC D E FG

Then it is impoſſible I can obſerve that law any otherwiſe, than either by putting F up towards G with a ſharp, as in *n.* 2. Or elſe bringing B down towards A with a flat, *n.* 3.

1. A BC D EF G A BC D EF G &c.
2. AB C D EF G AB C D EF G

In the firſt caſe (*n.* 2.) we have 3 notes, and 2 alternately, which anſwers one property; then we ſhall find the ſharp key to be in G, becauſe there is a half note below it and two whole notes above it, which anſwers another property. See Prop. IV. Cor. 1. and Prop. V. Cor. 1. and 2.

In the ſecond caſe (*n.* 3.) we have alſo 2 and 3 notes alternately; and the ſharp key will now be in F, becauſe there is a half note below it, and two whole notes above it. And it is impoſſible in either caſe, that the key can have any other ſituation.

What is here ſaid of the ſharp key, is equally true of the flat key, becauſe it is always a note and a half below the ſharp key. For if A be the flat key (*n.* 1.) then E will be the flat key (*n.* 2); for according to its property, it has a whole note above it, and two whole notes below it. And (*n.* 3.) where B is flat, the key will be in D; for there alſo it has one whole note above it, and two below it.

Fig. This foundation being laid, I ſhall proceed to ſhew where the key falls for one, two, three, four, &c. flats or ſharps, being given. And on the contrary, the key being given; to find what flats or ſharps it muſt have: *and firſt for the ſharps.*

1. A BC D EF G A BC D EF G &c.
2. A BC D E FG A BC D E FG &c.
3. A B CD E FG A B CD E FG &c.
4. A B CD E F GA B CD E F G &c.
5. A B C DE F GA B C DE F G &c.
6. AB C DE F G AB C DE F G &c.
7. AB C D EF G AB C D EF G &c.

And ſo on till all the notes be ſharped, but the practice never goes ſo far; but it ſerves to ſhew how nature directs the whole proceſs.

N. 1. is the natural ſcale for either key, no matter which; they are now in A and C. All the reſt follow from one another, thus;

N. 2. F muſt be ſharp, and the keys fall in E and G.

N. 3. C muſt be ſharp, and the keys are in B and D.

N. 4. G muſt be ſharp, and the keys are in F♯ and A.

N. 5. D muſt be ſharp, and the keys fall in C♯ and E.

N. 6. A muſt be ſharp, and the keys are in G♯ and B.

N. 7. E muſt be ſharp, and the keys are in D♯ and F♯.

Here it is evident the notes muſt be ſharped in this order only, F, C, G, D, A, E, &c. *Again for the flats.*

1. A BC D EF G A BC D EF G &c.
2. AB C D EF G AB C D EF G &c.
3. AB C DE F G AB C DE F G &c.
4. A B C DE F GA B C DE F G &c.
5. A B CD E F GA B CD E F G &c.
6. A B CD E FG A B CD E FG &c.
7. A BC D E FG A BC D E FG &c.

N. 1.

N. 1. is the natural ſcale, the keys being in A and C; and each number follows from the foregoing one. Fig.

N. 2. B muſt be flat, and the keys fall in D and F.

N. 3. E muſt be flat, and the keys fall in G and B♭.

N. 4. A muſt be flat, and the keys are in C and E♭.

N. 5. D muſt be flat, and the keys are in F and A♭.

N. 6. G muſt be flat, and the keys are in B♭ and D♭.

N. 7. C muſt be flat, and the keys fall in E♭ and G♭.

Here it is plain the notes muſt be flatted in the order B, E, A, D, G, C, &c. and in no other.

It has now been ſhewn where the key falls, for any number of flats and ſharps. Now I ſhall ſhew how to find what flats or ſharps belong to a tune, when the key is given; or which is the ſame thing, when a tune is to be tranſpoſed ſo many notes higher or lower as one ſhall chooſe. And this is to be done by the reverſe method, and is beſt ſhewn by examples.

Suppoſe I have a tune with two flats, and I want to tranſpoſe it to have no flats. Look in the laſt catalogue for two flats, and the flat key will be in G; but where there are no flats, the ſame key is in A, which is a note higher. Therefore the tune muſt be ſet a note higher. But ſuppoſe you want to ſet it ſo many notes higher or lower, as ſuppoſe two notes lower. Look for two flats in the latter catalogue, and you will find the key in G; then to have it two notes lower, the ſame key muſt be in E; therefore looking for E, you will find it with 6 flats; but this will not anſwer the purpoſe; therefore look for E (the flat key) among the ſharps (in the firſt catalogue); and you will find it to have one ſharp, which muſt be in F; ſo that the tune may be ſet two notes lower by putting F ſharp.

Fig. Again, if a tune has 3 ſharps, and I want it with only one. With 3 ſharps, the ſharp key will be in A; and with one ſharp in G; ſo the tune muſt be ſet a note lower. But if I want it two notes lower, then the ſame key muſt be in F, which requires 6 ſharps, which will do nothing; therefore we muſt take one flat, to have it in F.

73. But the general ſcale, derived from the foregoing proceſs, will be more expeditious. By this any tune may be ſet as many notes higher or lower, as we pleaſe; or to have as many flats or ſharps as we will, not exceeding four.

To ſet a tune higher or lower. Seek the flats or ſharps given in your tune, and obſerve the note in the ſame bar. Then find the note that is ſo much higher or lower; and the flats or ſharps that are joined with it, muſt be ſet to the tune.

To ſet a tune to have any number of flats or ſharps. Find the flats or ſharps given in the tune, and then the flats or ſharps to be ſet to it. Then the two correſpondent notes will ſhew how much higher or lower the tune is to be pricked.

For example. Suppoſe a tune that has 2 flats, is to be ſet 2 notes lower. Here G is the note along with 2 flats, therefore E (which is two notes lower) is the note ſought, with which there is one ſharp; which anſwers the queſtion.

Again, if a tune with 3 ſharps is to be tranſpoſed to have but one. Find 3 ſharps, which is at F; then find one ſharp, which is at E, and that is one note lower than F. Therefore the tune is to be pricked one note lower. But if I want it 2 notes lower, find D, which gives one flat.

Many Corols. may be drawn from this Prop. I ſhall mention ſome.

Cor. 1. *Hence any tune may be tranſpoſed, ſo as to have either flats or ſharps upon it. And therefore,*

Cor.

Cor. 2. *A tune is not ſaid to be on a flat or ſharp key, becauſe of the flats or ſharps ſet before it. But it is ſaid to be in a flat key, when it has the leſſer third above it; and in a ſharp key when it has the greater third.* Fig. 73.

Cor. 3. *If a tune requires a ſingle ſharp, it can be no way but in* F. *If it requires two, they muſt be in* F *and* C. *If three, in* F, C, *and* G, *&c. and no way elſe.*

Cor. 4. *If a tune requires one flat, it muſt neceſſarily be in* B. *If it requires two, they muſt be in* B *and* E. *If three, in* B, E, *and* A, *&c. and no where elſe.*

Cor. 5. *In modulation, or paſſing from one key to another; the muſician muſt play ſuch flats and ſharps as are proper to this laſt key, while he continues in it; omitting the proper notes.*

For playing in another key, is juſt the ſame thing, as when a perſon has been ſinging a-while; and then another perſon with a voice ſo much ſmaller, ſhould begin to ſing ſome diviſion of the ſame tune. It is evident he muſt keep the ſame air, *i. e.* obſerve the ſame flats or ſharps.

Cor. 6. *Let a tune be tranſpoſed how it may, ſtill there can be no more notes played but ſeven in an octave.*

For in the natural ſcale there are but 7 notes; and there are no more when tranſpoſed; though ſome notes are changed for others, flat or ſharp.

PROP. X.

To explain the gamut, or the method of writing down notes for the practice of muſic.

I have already demonſtrated the natural conſtitution of the ſcale of muſical notes, how many

Fig. notes and half notes are in an octave, and in what order they are placed. I am now to ſhew how they muſt be written down, ſo as every written note may be diſtinctly known, and may fitly repreſent the ſeveral ſounds they ſtand for; ſo that the practitioner may readily play them when he ſees them upon paper, which is the deſign and perfection of this art.

The gamut for any ſingle part of muſic is made by writing the notes upon a ſtaff of five lines, formerly they uſed ſix lines; and giving names to every line and every ſpace; and upon theſe lines and ſpaces all the different notes are written in or-
74. der. But to take in the whole compaſs of muſic. The ancients invented a gamut conſiſting of eleven lines, as PQ, and as many additional lines, as they had occaſion for, above or below; and upon this ſtave of lines, they writ all the notes that can be uſed in muſic, as *g*, *a*, *b*, *c*, *d*, &c. The firſt, or higheſt ſtave of five lines RS, they allotted to the treble; the loweſt five lines TV to the baſe; and the middle line, and two on each ſide, to the tenor: and upon each part, they placed the three cliffs to diſtinguiſh them; and placed the notes in all the lines and ſpaces, with their names. Theſe cliffs, and the names placed where the notes are to ſtand, are ſet down in the figure. And this ſcale is uſed to this day.

But certainly the inventors were very unlucky in forming ſuch a gamut, and the moderns are more ſo, for continuing the uſe of it. There is no manner of advantage by having it in this form; for no body ever played a ſingle part, off a ſcale of eleven lines. For muſic is always divided into parts, and each part is ſeparately pricked upon five lines. But the diſadvantage of this ſcale is enormous, and is the worſt ſcheme that any body could have fallen into. For taking all the parts of it,

cor-

correſponding to the ſeveral parts of muſic, and there are no two parts of it alike. For the treble five lines has the higheſt line F; the baſe five lines has the higheſt line A; the tenor five lines has the higheſt line G, and ſometimes B and E, according as the tenor cliff is placed, which by this contrivance muſt always be C. Nay, it is ſometimes placed upon the higheſt and loweſt lines. The conſequence of all this is, that whoever would learn to play upon the harpſichord or ſpinet, muſt be obliged to learn 5, 6, or 7, different gamuts, when one might have done the buſineſs. But who can ever have the patience, or ſpare the time to do this; certainly none but ſuch as have nothing elſe to do, or make it their buſineſs. Now had but the inventors made their ſcale of 12 lines inſtead of 11, and had taken the loweſt five for the baſe; it would have been uniform with the treble; and then there would only have been one gamut to learn, inſtead of two, now in uſe, for theſe two parts, the treble and baſe, which are the principal parts. But they had not even the good fortune to do this. Would not any body wonder at the patience of a perſon, that for variety, ſhould make a new alphabet, by making *b* ſound as *a*, *c* as *b*, *d* as *c*, &c. and ſo learn to write and read that way; any perſon acquainted with the preſent alphabet, would hardly think that the word *pof* was deſigned to expreſs *one*. This ſcheme of five or ſix different gamuts is of a ſimilar nature. But I am not the firſt that has taken notice of this abſurdity. Fig. 74. 75.

It is certain, that the authors of this confuſed and intricate ſcheme had never conſulted nature, which in all phyſical enquiries ought to be our chief guide; if they had, ſhe could have pointed out a plain and direct method, which would have prevented the infinite perplexity which a learner is thrown into, by ſo many inconſiſtent and contradictory

Fig. 75. dictory methods of notation. For can there be a greater abſurdity, than to make the ſame line or ſpace, ſometimes to ſtand for one note or letter, and ſometimes for another, even through the whole alphabet of notes. And I have often wondered, that they did not even uſe different letters for the correſpondent notes in different octaves. And this ſhews that muſic has not met with the ſame improvements, that other arts and ſciences have done.

I ſhall now deſcribe that method of notation which nature has laid down, and which prevents all that nonſenſical inconſiſtency, confuſion, and puzzling, which has been ſo juſtly complained of. It is acknowledged by all perſons that know any thing of muſic, that all octaves are the ſame, and have all the ſame properties, and the ſame ſimilitude throughout; all octaves then, being ſo near akin, ought to be ſo pricked down or written, that the ſchemes or draughts may have the ſame ſimilitude or agreement, as their originals. Therefore
76. let the higheſt octave, or the treble part be ſet upon a ſtave of five lines, with the higheſt line F, as we have it now. Let the ſecond octave or tenor be
77. alſo ſet upon another ſtave of five lines, and let the higheſt line be F, as before, but an eighth be-
78. low the firſt. Laſtly, let the loweſt octave or baſe be ſet upon another ſtave of five lines, and the higheſt line F, which is two eighths below the firſt. Then let the proper cliffs be ſet upon each; the treble cliff upon G, the tenor cliff upon C, the baſe cliff upon F. So there will be but one gamut, or one order of notes, for all three; each an eighth below the other. I make no alterations with the treble, but make it the ſame as before, becauſe it is univerſally known; and therefore I accommodate the reſt to that, which are but known to few. I deſign each of theſe to be further extended than five lines, by drawing occaſionally other lines either

ther above or below, as is practiſed now. This method is agreeable to the nature of things, conſiſtent with itſelf, and the eaſieſt that can be thought of. And I will venture to ſay, that whoever has a deſire to play upon the harpſichord, will learn in a quarter of the time, that the common way requires. And that many people are deterred from learning, from the apparent difficulties in the common formidable ſcheme. Fig. 78.

The only objection I can think of, is this; all our muſic is written according to the vulgar method; and in that form can be of no uſe. But that is eaſily remedied, by tranſpoſing it, as occaſion requires; as I have often done myſelf; which will not be a great deal of trouble; and will, beſides, teach the ſcholar to write muſic.

I do not mean by this, that any perſon who has had pains and patience enough to learn the common way, ſhould lay it aſide to take this; for he need not have two methods, to do one thing. Yet thoſe perſons, if they know the common way never ſo well, would fall into this method, without the leaſt trouble or difficulty. And I believe if this method was once begun, it would ſoon become univerſal. You have this gamut in Fig. 82. where A, B, C, D, &c. are ſet to the notes belonging to them, and whoſe places are invariable. I have not put the names at length, A lamire, B fabemi, C ſolfant, &c. becauſe the letters A, B, C, &c. ſerve my turn as well. 82.

There is a like abſurd practice made uſe of in ſinging; which is, that when B is flat, then *Mi* (whoſe true place is B), and the whole train of notes attending it, are to change their places; and *Mi* is to fall in E. Then if a flat happens in E, they are all to change their places again, and *Mi* muſt fall in A, and ſo on for more flats. As if *Mi*, or any other ſyllable, could not be applied to any

Fig. any ſound, or muſical note, flat or ſharp. A method ſo abſurd and irrational, that no reaſon can be given for ſuch a practice, but the humour of making every thing as obſcure and difficult as it can poſſibly be made. And ſome do not ſcruple to ſay, that muſic has been deſignedly involved in all this darkneſs and obſcurity by the profeſſors of this art, in order to keep the ſecret to themſelves; for it is certain they go the furtheſt way about. As to ſinging, there can be no occaſion for any names for the ſeven notes, but the ſeven letters A, B, C, D, E, F, whoſe places are eternally fixt. What then is all this pother about? Such are the inconſiſtencies men run into, when they forſake the plain paths of nature, to follow precarious methods, that have no foundation in the nature of things.

PROP. XI.

To deſcribe the ſeveral ſorts of notes uſed in muſic, and other marks and characters uſed therein; and of time.

79. *Notes* are of ſeveral forms, according to the lengths of the ſounds they are to expreſs. A a *brief*, B a *ſemi brief*, C a *minnim*, D a *crotchet*, E *quavers*, F *ſemi-quavers*, G *demi-ſemi-quavers*. Their length in time of playing is, a brief (now out of uſe) two ſemi-briefs. A ſemi-brief, two minums. A minnim, two crotchets. A crotchet, two quavers. A quaver, two ſemi-quavers. A ſemi-quaver, two demi-ſemi-quavers; their tails are often tied together, for the benefit of the eye. A *prick* ſet after a note makes it half as long again. The heads of the notes muſt ſtand in the proper places; the tails may be turned any way for convenience.

80. *Reſts* are marks to ſhew, that the muſic ſtops for a certain length of time, expreſſed by the length of

of ſome note. The reſts are ſet along with the notes of the ſame length. Fig.

A *flat* (A) is a mark ſet before a note, to ſhew it is to be played half a note lower. 81.

A *ſharp* (B) ſhews, that the note before which it is ſet, is to be played half a note higher.

A *natural* (C) ſet before a note, ſhews the note muſt be played without flat or ſharp.

When any flats or ſharps are ſet at the beginning of a leſſon, every note upon ſuch lines and ſpaces muſt accordingly be played flat or ſharp throughout; and all their octaves.

A *repeat* (A) ſhews, that the tune is to be played over again from that mark to the end of the ſtrain, or to the *cloſe* E. 83.

A *direct* (B) placed at the end of a ſtave, ſhews what note begins the next ſtave.

A *ſhake* (C) is performed by ſhaking the note above.

A *beat* (D) is performed by ſhaking the note below.

A *ſtave, or ſtaff*, is the five lines drawn to write the muſic on, as in Fig. 76, 77, &c. The additional lines above or below are called *ledger-lines*.

Bars, are lines drawn acroſs the ſtave; a ſingle bar is a ſingle line; and theſe go quite through the tune, and are to divide the time equally through the whole part. For every bar (contained between one croſs line and another) is of the ſame length, and muſt conſiſt of the ſame number of any one ſort of notes. The double bars have two croſs lines, and are only ſet at the end of a ſtrain, or part of the muſic.

Time, is that kind of movement which is proper to the tune, and is of two kinds, common time and tripple time.

Common time, is when the notes move by two and two. In a tune, or leſſon in common time, a bar

Fig. 84. bar contains the length of ſome particular note, as a ſemi-brief, minum, &c. which may be divided at pleaſure. There are ſeveral moods in common time, denoting a ſwifter or a ſlower motion. The ſloweſt is marked as you ſee at A; the next marked at B is faſter; third at C is the quickeſt; ſometimes they are marked as at D and E, and are of the ſame meaſure as C, and have but two crotchets in a bar, while the reſt have four. The length of a bar in the ſwifteſt (C) is a ſecond; and in the ſloweſt, two ſeconds.

Tripple time; here the motion is by three and three; and there are always three in a bar of ſome ſort of notes; or twice three, or ſome number of three's. The moods are as follow.

The ſloweſt kind is marked thus $\frac{3}{2}$, and has 3 minums in a bar, the next is 3 or $\frac{3}{4}$; here is a pricked minum in a bar.

The quickeſt is marked $\frac{3}{8}$; here is a pricked crotchet in a bar. There are alſo ſome compound moods, as $\frac{6}{4}$ and $\frac{6}{8}$; here the crotchets or quavers go by 3 doubled, that is, two three's in a bar.

Alſo in the moods $\frac{9}{4}$ and $\frac{9}{8}$, the crotchets or quavers go by 3 trebled, that is, by three three's in a bar.

Laſtly, there is the mood $\frac{12}{8}$, in which the quavers go by 3 quadrupled, that is, by four three's in a bar.

A general rule to know the movement belonging to any of theſe moods, or how it is barred. Let a ſemi-brief be the integer, and then the fraction denoting the mood, will tell you what part or parts of it are in one bar. Or which is the ſame thing, take the part of a ſemi-brief denoted by the lower number; and the upper number will tell you how many of theſe parts are in a bar. Thus in $\frac{9}{4}$, the 4th part of a ſemi-brief is a crotchet; therefore there are 9 crotchets in a bar.

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ca. 390
UNIV. OF MICH.

The diſtinguiſhing mark of common time is, Fig. that all notes go by two's; and that of tripple time, that ſome particular note goes by three's, whilſt all the reſt go by two's. And in all ſorts of time, the movement is compleated at the end of every bar; at which, there is a kind of intermiſſion or ſtop, which divides the tune regularly into ſimilar parts or elements. And it is probable, that the compound moods in triple time have been contrived to anſwer ſome ſorts of meaſures in poetry, or verſe; as we have ſongs in many different ſorts of meaſures, or number of ſyllables in a verſe. And theſe moods have been fitted to anſwer them.

There are ſeveral other marks uſed in muſic, but theſe are the principal.

PROP. XII.

To explain the phænomena of communicating ſounds from one ſtring to another.

1. If two ſtrings of any inſtrument be tuned uniſon, and one of them ſtruck, and made to ſound; the other will give a ſound alſo. Or if the ſtrings be upon different inſtruments, the effect will be the ſame, if they are placed near together.

The reaſon of this is, that if the ſtring AB be 85. ſtruck, and made to vibrate, the other ſtring, though at reſt at firſt, will receive all theſe impulſes of the air, which muſt neceſſarily put it into motion; for every force acting upon a body at liberty, will excite a certain degree of motion; and this muſt be the ſame vibrating motion which the air has; and that is the ſame it receives from the vibrating ſtring. And having the ſame vibrating motion, it muſt give the ſame ſound. This is the caſe with the drum of the ear, when it receives the impreſſion from the tremulous air, and cauſes the ſound

Fig. 85. found to be heard. For as the vibrating motion of a ſtring communicates a like motion to the air; ſo, *vice verſa*, the vibrating motion of the air communicates a motion to a ſtring, that is, in a proper diſpoſition to receive it.

86. 2. If two ſtrings be tuned to an octave, and the higher ſtring be ſtruck, the lower will give the ſame ſound; as may be tried on a fiddle. If AB be an octave above AC, and AB ſtruck, then AC will give the ſame ſound. But then it is not the whole of the ſtring AC that trembles, but the two equal parts AD, DC, which are in uniſon with the firſt AB. The reaſon here is the ſame, for the impulſes of air from the ſtring AB, cannot give the ſame motion to the whole ſtring AC, becauſe it can naturally vibrate only half as faſt, and therefore one impulſe deſtroys the effect of another, for producing this motion. Therefore the effect of theſe ſeveral impulſes will at laſt be this, that the ſtring will be forced into two vibrations of the two equal parts AD, DC; which are in uniſon with AB. For the middle point is at reſt, for a ſmall piece of paper laid on D, will be unmoved; but ſhifting it gradually towards either end, it will be thrown off, or made to tremble.

87. Likewiſe if the ſtring EF be a twelfth below AB, or a fifth below AC, and EF be divided into 3 equal parts, theſe three parts will vibrate, whilſt the points of diviſion are at reſt. For the vibrations of the air, acting on the ſtring, will move the parts ſeparately, being each in uniſon with AB. And if EF was two eights below AB, then the four parts of EF would move; but the motion at laſt grows ſo languid and faint, that it cannot be perceived.

But if any of theſe ſtrings are ſtruck at the points of diviſion, it will only give a confuſed ſound; for the motions are then inconſiſtent, by reaſon the points

points are diſturbed that ſhould be at reſt. A pipe will alſo cauſe the like vibrations and ſound in a ſtring, that is uniſon with it. Fig.

3. If a ſtring be tuned a perfect twelfth to the baſe, and another ſtring a 17th to it, or a greater ſixth to the other note. And if the baſe be ſtruck, the two others will ſound; but it is only a good ear can perceive them. I take it for granted, that the impulſes of the air acting equally on both ſides of a ſtring, will give it no motion; but acting more on one ſide than the other, will give it a motion proportional to that exceſs. Here in the firſt caſe, the lengths of the ſtrings are 1 and $\frac{1}{3}$, therefore the upper ſtring vibrates 3 times for the lower once. Divide the time into 3 equal parts, each equal to one vibration of the upper ſtring; and let the vibrations of each, towards the right and left, be denoted by *r* and *l*, and their conſpiring or oppoſing one another, that is, the addition or diminution of motion of the upper ſtring, by *a* and *d*. Then their motions will ſtand thus,

lower	*rrr*	*lll*	*rrr*	*lll*
upper	*rlr*	*lrl*	*rlr*	*lrl*
	ada	*ada*	*ada*	*ada*

In this compariſon, it appears there are 2 *a*'s for 1 *d*; and conſequently $\frac{1}{3}$ of the vibrations conſpire to move the upper ſtring, the remaining two deſtroy one another. Therefore the upper ſtring will be made to vibrate, by the vibrations of the lower.

The other ſtring (the 17th) is $\frac{1}{5}$ in length, which will give 5 vibrations for one of the baſe. Then noting down as before, which way theſe vibrations (or parts of vibrations) lie; they will ſtand thus,

lower	*rrrrr*	*lllll*	*rrrrr*	*lllll*
upper	*rlrlr*	*lrlrl*	*rlrlr*	*lrlrl*
	adada	*adada*	*adada*	*adada*

By

Fig. By this it appears, there are 3 *a*'s for 2 *d*'s, therefore $\frac{1}{5}$ part is ſpent in giving motion to the upper ſtring; for it is certain, where the motions conſpire (the *a*'s), it will be accelerated, and where they oppoſe (the *d*'s), it will be retarded. And in the whole, it will be made to vibrate. The ſtring $\frac{1}{7}$ will alſo be made to vibrate a little; but all of them give ſo ſmall a ſound, as only a good ear can perceive. But none beſides, as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, &c. can be made to vibrate upon this account, from the vibrations of the baſe; as may be eaſily computed after the ſame manner.

There are many more ſuch phænomena of muſical ſounds, which my room will not admit of. I ſhall only obſerve, that ſome ſounds excite others as has been ſhewn, and theſe again create others, and theſe produce more; and all by ſetting ſuch ſtrings a vibrating as are in a diſpoſition to receive the motion, that is, whoſe vibrations any way fall in with the others. So that at laſt there is a ſtrange medley of muſical ſounds, chiming one with another, as may be obſerved in any full ſtrung inſtrument of muſic.

PROP. XIII.

To lay down the general rules of taking concords and diſcords in compoſition.

It has been ſhewn before what notes ſounding together are concords, and what are diſcords; we muſt now ſhew how they muſt be put together ſo as to make the beſt harmony. For if they be put together at random, they will but make indifferent ſort of muſic; therefore muſicians have, by trials and experiments, found out and laid down rules how they are to be managed, to make the muſic as ſweet as poſſible. In the firſt place a baſe is to be

be framed, which muſt expreſs the air of the tune, and muſt move, not ſo much by degrees, as by leaps of a 3d, 4th, or 5th. It muſt have proper cloſes, and end at laſt in its proper key. Then the other parts are made from this, by taking to each note thereof its proper concord, according to the following rules. But ſometimes a treble is firſt made, and a baſe put to it; and then you muſt reckon your concords downwards, as before you did upwards from the baſe. In compoſition, good air as well as good harmony ought to be maintained quite through the piece, to which all ſtrict rules muſt give way. For in all caſes the ear muſt be umpire of the ſweetneſs of the muſic, which ought to be attended to before any ſervile fixt rules. The art of compoſing muſic in parts is called *deſcant*, but formerly *counterpoint*, whoſe rules here follow. In compoſing, ſtrive to make one part imitate another in the notes. And in taking the concords, note muſt be ſet againſt note, according to the following rules; and the parts may be allowed to paſs through one another. And in muſic of many parts, the upper parts ought to be joined together as cloſe as poſſible. Fig.

Paſſages of Concords.

Rule 1. Two fifths or two eighths are not allowed to go together, in any two parts; for they are too luſcious for the ear; and playing an eighth to the whole tune may as well be admitted. Yet they may be allowed when one part ſtands ſtill, or riſes or falls an eighth, or when they paſs by contrary motion.

Rule 2. One may take as many concords (of one ſort) as one will, the baſe ſtanding ſtill. Or may move to any concords.

Rule 3. One may change from any one, to any other concord; but more elegantly to ſome than

Fig. others. Not ſo well from one perfect to another. Nor when the higher part ſkips to a perfect, and the baſe moves but a note. Nor when both ſkip to a perfect. Nor to riſe with the baſe from a ſixth to an eighth, and the contrary.

Rule 4. You may take as many thirds or ſixths together as you will, eſpecially if they are alternately greater and leſs. For the muſic is the beſt, where there is the moſt variety of concords.

Rule 5. Take ſuch concords as are had with the leaſt remove. Therefore the leſſer ſixth paſſes beſt into a fifth, and the greater ſixth into an eighth.

Rule 6. Moving by leaps is beſt done into imperfect concords.

Rule 7. In a ſharp key, a ſixth is required to the ſecond and third, both above and below the key, becauſe the baſe commonly wants a third to its full compaſs.

Rule 8. When the parts move gradually, one may take two or three thirds and ſixths, or a fifth and ſixth by way of binding.

Rule 9. In compoſing ſeveral parts, ſet not the ſame cord twice over, or between any two different parts. For they muſt vary thus, 3d, 5th, 8th; or 4th, 6th, 8th.

Rule 10. When a paſſage is diſputed, the ear muſt be judge. For by that *criterion* were all the rules found out.

Paſſages of Diſcords.

Rule 1. Diſcords are allowed in tranſition or breaking of notes; provided the leading note be a concord.

Rule 2. Diſcords are allowed in *binding notes*; that is, when a note of one part breaks off in the middle

middle of a note of another part. Theſe are cal- Fig.
led notes of *ſyncopation*, or *driving notes*. Here a diſcord is allowed in half the binding note.

Rule 3. Diſcords muſt only be upon ſhort notes, that ſoon paſs the ear.

Rule 4. Diſcords are allowed when the baſe ſtands ſtill, and the other parts move gradually; here they muſt needs fall upon diſcords.

Rule 5. Diſcords are allowed at a cloſe, and are generally reſolved into the neareſt concords. Diſcords are prepared by concords, and reſolved into concords. Thus a leſſer 7th is reſolved into a fifth.

Although diſcords are diſagreeable in themſelves, yet when they paſs into concords it makes the muſic very delightful; and is like a ſweet cordial after a bitter potion.

Cadences.

Rule 1. The baſe intends a cloſe, when it falls a fifth, after riſing or falling one or two notes. And in falling a fifth (or riſing a fourth), the former note requires a greater third. At a cloſe the baſe is to fall a fifth.

Rule 2. The cloſes in any key, are the key, the 5th, the 2d, and the 6th and 3d.

Rule 3. The cloſes in any key, fell naturally in the order ſet down in the laſt rule.

Rule 4. At a cloſe, you may remove by leap to any concord.

Rule 5. At a cloſe, diſcords are uſed both by paſs, and binding, promiſcuouſly according to the former rules.

This mixture of concords and diſcords do very much delight the ear, and keep it in ſuſpence, till a perfect concord comes to compleat the harmony.

Though the cloſes naturally come in due courſe one after another, yet a judicious muſician, by the

Fig. artful introducing flats and ſharps, can ſkip over ſome and take others out of courſe; but then it is a little painful to the ear; and ſo, indeed, in ſome degree, is every ſhifting of the key.

There are ſeveral particular rules of the paſſages of concords and diſcords which I have not mentioned, but they all depend on what has been delivered.

In playing *figured* or *thorough baſe*, where nothing is marked, *common cords* muſt be played, which are the 3d, 5th, and 8th. The other cords are marked with their proper figures. A flat or ſharp ſet among the figures, ſignifies a flat or ſharp third to be played. To play a good thorough baſe, and manage diſcords rightly, one muſt know, 1. How to prepare them. 2. How to accompany them. 3. How to reſolve them. Accompaniments are cords taken in, to fill up the harmony. But I have no deſign of meddling with the practical part of muſic; I have treated of it ſo far as mathematics have any concern in it, and I promiſed no more.

ART.

Fig.

ART. XVI.

A general Method of Philoſophiſing; or, Rules to be obſerved in Philoſophical Enquiries.

THE deſign of philoſophy is to find out the nature and cauſes of things from their viſible effects; and theſe effects are the phœnomena which are made known to us by obſervation and experiments. In order to philoſophize aright.

1. We muſt make a competent number of obſervations and experiments, whereby we may know that the effect we reaſon about is certainly true.

2. We muſt find out all the cauſes, as far as poſſibly we can, that can produce this effect; and the truth of them muſt be mathematically demonſtrated, when neceſſary. This is the moſt difficult part of the buſineſs; for it is hard to know when we have got all theſe cauſes. And therefore a philoſopher ought to have a very extenſive view of things, and a thorough knowledge of the ſubject he enquires after. If we be not ſure that we have enumerated all the cauſes, then the effect may proceed from ſome cauſe we have omitted; and ſo far the propoſition will remain doubtful.

3. Sometimes it is neceſſary to ſhew and demonſtrate what cauſe cannot produce the effect; eſpecially when that effect has been vulgarly and erroneouſly attributed to ſuch imaginary cauſe.

4. To proceed, we muſt compare all the poſſible cauſes of any two of theſe effects (or phœnomena) and reſerve ſuch as are common to both. Again, compare theſe with all the poſſible cauſes of a third phœnomenon, and reſerve the cauſes that are common to both. In like manner compare theſe with

Fig. all the cauſes of a fourth phœnomenon, &c. Proceed thus till there be but one cauſe at laſt common to both; then it is evident this is the cauſe ſought after, for all the reſt are exhauſted.

5. After the true cauſe is found out, the ſame principle muſt be applied to the ſolving all phœnomena of the ſame kind; and if the explications given thereby thoroughly agree, it will confirm the truth of the principle from whence theſe ſolutions are had. For if they be true, they muſt ſtand the rigour of mathematical calculation. But if they cannot do this, we muſt take it for granted, that we have fallen upon ſome wrong principle; and therefore we muſt look out for ſome other. That this coincides with the Newtonian method of proceeding, I ſhall ſhew in the following Examples.

EXAM. I.

To find the cauſe of the revolution of the heavenly bodies, and of retaining them in their orbits.

Phœnomena.

1. The ſatellites of Jupiter and Saturn, deſcribe areas proportional to the times of deſcription about their primary planets.

2. The primary planets deſcribe areas proportional to the times of deſcription, about the ſun; but not about the earth.

3. The periodic times of the ſatellites about the primary planets, or of the primary planets about the ſun, are in the ſeſquiplicate ratio of the mean diſtances from their reſpective centers.

4. The moon deſcribes areas round the earth, proportional to the times of deſcription nearly.

Propoſitions.

1. Every body that deſcribes areas proportional to the times of deſcription about a point,

is

is urged by a centripetal force tending to that point. Fig.

2. If the periodic times of revolving bodies are in the ſeſquiplicate ratio of the radii; the centripetal forces will be reciprocally in the duplicate ratio of the radii.

3. If a body moving in a fluid revolve round a center, it will be retarded, and the area will not be as the time.

4. A ſpherical body revolving round its axis in an immenſe fluid, will cauſe the parts of the fluid to revolve, ſo as the periodic times will be as the ſquares of the diſtances, from the center of the ſphere. And this will conſtitute a vortex; and bodies revolving in it muſt be of the ſame denſity, and obſerve the ſame laws of revolution.

5. If the moon is retained in its orbit by a force reciprocally as the ſquares of the diſtances. That very force at the earth's ſurface will make bodies deſcend 16.1 feet in a ſecond.

Solution.

From phœnomena 1ſt and 2d, the revolving body muſt be carried round, and continued in its orbit, either by ſome force perpetually acting towards the center, in free ſpace. Or elſe it muſt be carried round in a vortex. For it cannot move in a reſiſting medium by Prop. III.

And by phœn. 3d, the revolving body is retained in its orbit by a force acting towards the center, which is reciprocally as the ſquares of the diſtances, by Prop. II. For it cannot be carried in a vortex by Prop. IV.

By phœn. 4th, and Prop. I. the moon is retained in its orbit by a force acting towards the earth, which is reciprocally as the ſquare of its diſtance. And by Prop. V. this is the force by which bodies deſcend, called *gravity* or *weight*.

Fig. Now any other cauſes than theſe we have mentioned cannot be thought on. And therefore the true cauſe ſought is diſcovered to be a gravitating force, with which all bodies in the univerſe are endued. And therefore the planets, comets, and all the ſatellites are retained in their orbits by this force, and all move in free ſpace, or at leaſt in a medium ſo extremely rare, as to make no ſenſible difference in a long ſpace of time.

This cauſe or principle being found out, when duly applied, will explain all the phœnomena relating to the moon's motions, the preceſſion of the equinoxes, the motion of the ſea, &c. as may be ſeen in the Principia, Lib. III.

EXAM. 2.

To inveſtigate the cauſe of the refraction of light.

Phænomena.

1. If the ſun's rays come through a hole in a window ſhut, and be refracted by a glaſs priſm; an oblong image will be formed on a paper, or the oppoſite wall, conſiſting of ſeveral colours, whoſe length is perpendicular to the axis of the priſm.

2. If another priſm be placed behind the former, and with its axis perpendicular to it; it will refract the image ſideways, of the ſame length as before, but of no greater breadth.

3. If a ſmall part of the light of any one colour, be again refracted through one or more priſms, the image will always be round, and of the ſame colour as before.

Propoſitions.

1. If light conſiſts of heterogeneous parts, which are differently refrangible; being refracted through a priſm, it will paint an oblong image on the oppoſite wall.

2. If

2. If light conſiſts of homogeneous parts, and Fig.
paſs through a priſm, ſo that the angle of incidence and emergence may be equal in the two refracting planes; the image will be nearly circular.

3. If light be of ſuch a nature, that its rays will ſplit or dilate when refracted; then a priſm will refract the light into ſuch an image, that the breadth will be as great as the length; and this always, by all ſucceſſive refractions; and the image will be enlarged by every refraction.

Solution.

From phœn. 1 and 2, it appears, that light is made up of heterogeneous parts, which are differently refrangible. For it cannot conſiſt of homogeneous parts, by Prop. II. Nor does its rays ſplit or dilate, by Prop. III. Moreover from phœn. 3, that ſmall part of light, which continues of the ſame colour, muſt be homogeneous, by Prop. II. Hence light is diſcovered to conſiſt of heterogeneous parts all mixt together, but capable of being ſeparated from one another by refraction, into the compounding homogeneous parts. And the properties of theſe homogeneous parts are innate and eſſential to them, and cannot be altered by any reflections or refractions whatever. The nature of light thus found out, may be applied to explain other phœnomena of the like nature.

EXAM. 3.

To find out the cauſe of the aſcent of water in a ſyphon or pump.

Phœnomena.

1. If the end of a ſyringe be put in water, or any fluid, and the piſton drawn up, the fluid will follow and riſe with the piſton. Or if the mouth be applied to the top of a tube immerſed

in

Fig. in a fluid, ſucking at the top will draw the fluid up.

2. Cement a braſs cock in one end of a cloſe tube, exhauſt the air and ſhut the cock. Then put the end with the cock into the water, and open the cock, and the water will aſcend into the tube.

3. Cement a ſyringe in the mouth of a glaſs bottle, quite full of water; and draw up the piſton, but no water will follow.

Propoſitions.

1. If a vacuum be impoſſible in nature; then any fluid having free liberty, will move towards ſuch a vacuous place to prevent it.

2. If there be a greater preſſure upon one part of a fluid than on another; the fluid will move towards the part having the leaſt preſſure.

3. Whatever fluid is drawn or moved by the force of ſuction, will always be drawn or moved by that force when applied, if the way be clear.

Solution.

By phœn. 1 and 3, water aſcends up a tube either by the preſſure of an external fluid, or by nature's *fuga vacui*; for it cannot aſcend by ſuction, by Prop. III.

By phœn. 2, the water aſcends the tube by the preſſure of an external fluid; for it is not by any *fuga vacui*, by Prop. I.

Hence water aſcends into cyphons and pumps by the preſſure of ſome external fluid; and we know of no ſuch fluid but the air or atmoſphere.

EXAM. 4.

To find whether the diurnal revolution belongs to the ſun or to the earth.

Phænomenon.

Phænomenon. Fig.

The body of the earth is never in the plane of the ſun's diurnal orbit all the year; except at the equinoxes, when it paſſes through that plane.

Propoſition.

Every body moving round another body as its center, to which it is continually drawn or impelled; muſt move in a plane paſſing through that central body.

Solution.

Hence the apparent diurnal revolution of the ſun, proceeds from the real revolution of the earth round its axis. For if the ſun did really move round the earth; then would the earth always be in the plane of the ſun's orbit, by the Prop. But by the phœn. ſhe is not ſo. Therefore the ſun moves not round the earth.

The phœnomenon is proved thus, the ſhadow of the end of a gnomon erected on any plane, deſcribes always a curve line on that plane, except at the equinoxes.

EXAM. 5.

To find whether the annual revolution belongs to the earth or to the ſun.

Phænomena.

1. The ſun and planets don't deſcribe areas round the earth proportional to the times.

2. The periodic times of the ſun and planets round the earth, are not in the ſeſquiplicate ratio of their diſtances from the earth.

3. The earth and planets deſcribe areas round the ſun, which are as the times of deſcription.

4. The periodic times of the earth, and other planets round the ſun, are in the ſeſquiplicate ratio of their diſtances from the ſun.

Propoſitions.

Fig.

Propoſitions.

1. Bodies that deſcribe areas porportional to the times of deſcription, round another body; are urged by forces directed to that other body.

2. If ſeveral bodies revolve round another body, and their periodic times be in the ſeſquiplicate ratio of the diſtances, they are urged to that body by forces reciprocally proportional to the ſquares of the diſtances.

Solution.

By Prop. I. and phœn. 1, the planets don't revolve round the earth. But by phœn. 3, and Prop. I. the earth and planets revolve round the ſun. Alſo by phœn. 2, 4, and Prop. II. the earth and other planets are urged by forces, reciprocally as the ſquares of the diſtances from the ſun, but not from the earth. Therefore the annual motion belongs to the earth.

ART.

Fig.

ART. XVII.

Optics, being a Tranſlation of ſome Papers of Sir J. Newton's Optical Lectures, Part II. never Engliſhed before, p. 239, &c.

BEFORE I proceed to another kind of experiments, it is neceſſary that we look more particularly into the form of the coloured image; which the light coming in at a ſtrait round hole into a darkened room, and paſſing through a priſm, paints there; and carefully take the dimenſions of all the colours, and their diſtances from one another, and alſo the ſeveral degrees of refrangibility, correſponding to all kinds of rays.

It was ſhewn before, that when a priſm (whoſe vertical angle is about 63 degrees) projects the image at the diſtance of 22 feet, its length was $13\frac{5}{4}$ inches, and breadth $2\frac{1}{4}$. And therefore the centers of the outmoſt circles, of which diſpoſed in length, that image conſiſts, were diſtant $10\frac{3}{4}$ inches. Now to this diſtance or contracted longitude of the image it is proper to refer the other dimenſions thereof, becauſe they have no certain relation to its abſolute length (which depends on the magnitude of the compounding circles). And by this I could inveſtigate with more exactneſs the places where the colours were the moſt perfect in their kind; and their confines, falling upon a croſs paper, which I noted with a pen, and by many repeated obſervations of this ſort compared together; I at laſt gained theſe particular concluſions.

1. The blue and violet on one hand, and the green and red on the other; take up on each hand half the image; ſo that the confine of the green and

Fig. and blue (which I may call a ſea-green) lies in the middle.

2. The place where the leeky or brighteſt green appeared, divided the contracted length of the image in the ratio of 3 to 5; ſo that the length being divided into 8 parts, that green was 3 parts diſtant from the extremity of the red, and 5 parts from that of the purple.

3. The ſpace taken up by all the green, as far as the confines of the blue and yellow, was about the fixth part of the whole contracted image.

4. The confine of the blue and purple, or the moſt perfect indigo, was diſtant from the confine of red and yellow, or the moſt perfect orange, about $\frac{7}{12}$ of the whole contracted image.

5. Laſtly, this diſtance of the indigo and orange, was divided by the confine of green and blue in the ratio of 2 to 3; ſo that this confine, or the middle of the image, was diſtant from the indigo $\frac{14}{60}$ parts of the whole contracted length, and $\frac{21}{60}$ parts from the orange.

When I had obſerved theſe things as carefully as I could, not altogether truſting to my own ſenſes, but (on account of the great difficulty of preciſely diſtinguiſhing the confines of the colours, and the places of greateſt perfection) taking the judgment of other people; I drew out the dimenſions of the image according as I had found them; as may be
88. ſeen in the figure. That is, with the centers X and Y $10\frac{3}{4}$ inches diſtant, and with the ſemi-diameters $1\frac{5}{16}$ inches, I deſcribed two ſemi-circles APC and BTD, and drew the tangents AB, CD.

Then I divided the line XY (which I before called the contracted length of the image) into 60 equal parts, and made LY = 9, IY = 20, HY = 30, and FY = 44 parts thereof. And erecting perpendiculars at theſe points, I diſtinguiſhed the image into five parts, correſponding to the five

more

more eminent colours; the part PF ſhewing the ſpace of the violet, and FH the ſpace of the blue, &c. which being done, I projected the coloured light upon this figure, that it might again appear, whether every colour continued ſtill within the aſſigned limits; and when the whole image filled the whole figure, the ſeveral colours with their ſeveral parts very well agreed; and in the mean while, I obſerved the places in theſe ſpaces (which were marked out by points in this ſcheme) where the ſeveral colours in their kinds appeared full and brighteſt. Fig. 88.

Now it is evident there can be no other intervals of theſe places and limits, terminating the colours; although, by the methods often mentioned before, you never ſo much diminiſh the circles, which compoſe the image, keeping ſtill the ſame centers. And for this reaſon, that the heterogeneous rays will be more ſeparated, and the colours come out more ſimple. For ſince, in the very terminating right lines AB and CD, the colours are quite ſimple; and the colours in the middle of the image near the line XY, appear to be of the ſame kind with thoſe lying between that and the margin; reaſon tells us, that a mixture of heterogeneous rays does not ſenſibly change the place of any colour, ſince their coming from this ſide and that, they temper one another. Thus the green and purple rays are equivalent to the blue, and therefore when mixt with it, don't alter that colour; but it appears the ſame as if no purple rays had been mixt therewith. But here we muſt except the terminating circles AC, BD, where the temperament on the outſide gradually decays, and becomes of a full red, on one ſide, where it runs into the terminating circle. But if any one ſhould ſtick here, he may try new experiments, and contract the breadth of the image, that the circles may 89.

be

Fig. 89. be leſs, the reſt remaining; and I doubt not but every thing will anſwer.

But although the confines of the colours, fell in the lines paſſing through the points F, H, I, L; yet the places where they appear the moſt full and intenſe, do not always fall in the middle of the included ſpace; for the blue, which in its kind was the brighteſt, and no way tinged with purple, falls nearer F than H, and the intenſeſt yellow was a little nearer L than I. And thus the red and purple appeared intenſe a little nearer the centers X and Y, than to the other limits; only the green was brighteſt in the middle between the limits I and H. Whence appears the reaſon, that although the yellow and blue by their mixture compoſe a green, yet the red and green, by reaſon of their too great diſtance, don't well compoſe a yellow, nor the green and purple a blue. Therefore ſeeing the colours are more cloſe about the middle, ſo as between the yellow and red, as likewiſe between the blue and purple, there is about a third part more interval than between the green and yellow, or blue, terminated on both ſides; by this the image is elegantly diſtinguiſhed into parts, proportional among themſelves, containing five principal colours, and two more which we ſhall take in, and theſe are, orange between the red and yellow, and indigo between the blue and violet; and that becauſe next to theſe five principal colours, theſe two eminently appear, and the ſpaces aſſigned them are large enough for the degree of perfection of each; and thus the too great expanſion of the outer adjoining colours is ſhortened, and they are all made in a neater proportion to the quantity of the green.

Theſe colours being inſerted, I begun again to make obſervations, and (in a word) all things appeared, as if the parts of the image, which contain the colours, were proportional to a ſtring divided

vided, ſo as to give the ſound of the ſeveral degrees or notes in an octave. When I found this, I divided the figure of the image into like parts, as may be ſeen in the ſcheme. And then I tried how well the colours agreed with theſe parts. Thus, produce the contracted length of the image XY to Z, ſo that YZ = XY, and ſuppoſe the ſtring to be XZ, and that XY be ſo divided, that all the ſegments extended to Z, may ſound the ſeveral notes of an octave (*ſol*, *la*, *fa*, *ut*, *re*, *mi*, *fa*, *ſol*). And that is done by biſſecting XY in H, and triſecting it in G and I, and again triſecting XI in F, and taking KY a fifth part, and MY an eighth part of XY. And the ſemi-tones FG and KM will repreſent the indigo and orange; and the other five tones XF, GH, HI, IK, MY, the five eminent colours; all of which in particular, and the whole train of colours, falls exactly into the whole figure, and were contained reſpectively in the ſeveral parts. And about the middle of theſe parts, each colour in its kind appeared the brighteſt and moſt intenſe; even the purple and red, beyond P and T, abounded with light, gradually vaniſhing. Fig. 89.

But I could not obſerve theſe things ſo preciſely, but I muſt confeſs they may be contrived a little different. To which purpoſe, if there be taken eleven mean proportionals, between XZ and YZ, of which FZ is the ſecond, GZ the third, HZ the fifth, IZ the ſeventh, and KZ the ninth; then this diviſion of the image ſeems to agree very well with the expanſions of the colours. For ſuch minute differences as ariſe between this and the former diviſion, can hardly be accounted errors by the judgment of the moſt acute ſenſe.

How theſe diviſions differ will appear by the adjoining numbers, where the former expreſs the parts of the cord, divided in the muſical ratio; and

Fig. 89. the lower, the like parts, divided in a geometrical ratio. The card being divided into 360 parts.

360, 320, 300, 270, 240, 216, 202½, 180, (by muſical diviſion) and

360, 321, 303, 270, 240, 214, 202, 180, (by geometrical diviſion).

And I chuſed to lay down the former diviſion, not only becauſe it agrees very well with the phœnomena, but becauſe it may perhaps contain ſomething about the harmony of colours, analogous to the concords of ſounds, which the painters are not quite ignorant of, but I am unacquainted with. Wherefore it ſeems likely to a conſidering perſon, that the affinity which is between the outſide purple and the red, the extremities of the colours, is found likewiſe between the terms of an octave, which in a manner give the ſame ſound.

Hence at laſt the proportions of the ſines of refraction, belonging to all kinds of rays, are (mechanically) determined. Since in glaſs contiguous to air, the ſines of the extreme rays are as 68 and 69; divide the intermediate 1, in the ratio of the parts of this image, and there will ariſe 68, 68⅛, 68⅕, 68⅓, 68½, 68⅔, 68⅞, 69, for the ſines at the confines and limits of the ſeven colours, in reſpect of the common ſine of incidence 44½, when the refraction is made out of glaſs. But when it is made into glaſs, for theſe ſines take the numbers, 68, 68⅖, 68½, 68⅔, 68¾, 68⅘, 68⅞, 69; the common ſine of incidence being 106. And for the ſines for thoſe rays, where the colours are the moſt perfect in their kind, intermediate numbers may be found between theſe.

Thus in water contiguous to air, where the extreme ſines of refraction are as 90 to 91, one may find the intermediate ſines by a like diſſection of the intermediate unity (by making them as 90, 90⅛, 90⅕, &c. or 90, 90⅖, 90⅓, &c.) But note, theſe diviſions

diviſions are not preciſely geometrical, but yet as nearly accurate as any practice requires; and to toil any further, beſides the tedioufneſs of computation, would prove nothing but an affected and vain curioſity. Fig. 89.

There are other circumſtances concerning theſe colours, which I could now determine; ſuch as the various forms and expanſions thereof, for the various poſitions of the priſm revolving about its axis; or for the various refractive matter, of which the priſm is made, and with which it is encompaſſed; and alſo for the various magnitude of the vertical angle. But all theſe things will appear plain enough, from what is ſhewn in the firſt part, comparing them with the foregoing explications; ſo that all effects, as far as I know, may be found out, either for one ſingle refraction, or for more, and for any termination of light.

I have corrected a great many errors in this paper; how they have happened I do not know.

A tranſlation of part of a paper in the optical lectures, intituled thus. Of the phænomena of light paſſing through a refracting medium, terminated by parallel planes. Part II. Sect. 2. p. 261, &c.

Having gone through the phœnomena of triangular priſms, thoſe made by quadrangular ones, which are compoſed of parallel plains, come now properly to be ſpoken of. And I the more willingly engage in it, as the philoſophers have hitherto believed, that no colours can be generated after this manner, ſuppoſing that the latter ſurface deſtroys all the effects of the rays by a contrary refraction, to what the former produced; and they take it for a matter of experience, becauſe in glaſs windows, and ſuch like, they ſee none produced. But in this they are deceived, becauſe the quantity

Fig. and perfection of the colours, depends upon the distance of the parallel surfaces. For in glass plates, by reason of the small distance of the surfaces, the colours are very thin and weak, and contained in such a narrow space, as to escape the senses; but in glass of a greater thickness, or rather in glass vessels or parallelopipedons full of clear water, the colours are plainly seen to be generated.

90. For let ABCD be a glass, or water parallelopipedon, inclosed with air, whose parallel surfaces are denoted by the two lines AC and BD. And let the sun shine through it, passing obliquely thro' the small hole F. Then the rays, by the former surface at H, ought to be so unequally refracted, that they will diverge from one another, till they fall upon the latter surface at PT, and there paint all kinds of colours, as has been abundantly explained before. Now since by the parallelism of the refracting surfaces, the rays are only bent at the latter surface, just as much as they were at the former. It is necessary that they emerge in the air parallel to one another, and to the incident rays SH; and therefore the distances and positions they have acquired, they will preserve ad infinitum; and the colours produced will always remain the same without any variation. Wherefore if HP be the purple ray, and HT the red ray, refracted at H, the rays P*p*, T*t*, being again refracted, will emerge parallel to the incident rays SH, and to one another; and therefore they transfer the purple and red, which they give at P and T, to any distance unchanged, and preserve them every where without any variation; the purple being translated from P to *p*, and the red from T to *t*; and the rest of the rays into their correspondent intermediate places.

And this must exactly happen when the rays fall upon this prism in lines parallel to the same line SF,

SF, and they emerge parallel; but when they are inclined to one another, as happens when they proceed from different parts of the ſolar diſk; then indeed they emerge inclined to one another, and in their laſt tranſlations they undergo ſome change. Becauſe the circles affected with all kinds of rays, by which (being diſpoſed in length) the coloured image of the ſun is made, by falling on the ſurface BD; by reaſon of the diverging of the rays, croſſing one another in the hole at F, come out the more dilated, as the rays are further from the point of emergence; whilſt their centers, which are illuminated by rays from the ſun's center, moving in any direction before refraction, after refraction preſerve the ſame diſtances and poſitions for ever. And hence it is, that the ſpace *prt* illuminated by the ſolar light immitted into a dark room, is the more dilated by that, and drawn into an orbicular form, as its termination is further from the priſm; and the greenneſs in the middle R, if there is any, is gradually changed into whiteneſs; but if there is none but by the narrowneſs of the priſm, or wideneſs of the hole letting in the light, the whiteneſs poſſeſſes the middle of the colours; the ſame whiteneſs is ſenſibly dilated. But yet the colours are not thereby dilated, nor drawn into a leſſer ſpace, however leſs luminous they appear by the dilatation of the image. Fig. 90.

If we view any viſible objects through this parallelopipedon, they are no otherwiſe tinged with colours, than as a triangular priſm, eſpecially if the parallelopipedon be oblique enough to the tranſient rays, that it may refract much, and that the objects be ſufficiently near. For if the objects are far off, whether that diſtance lies between the parallelopipedon and objects, or the parallelopipedon and the eye; how great ſoever the refraction is, by the obliquity of the parallelopipedon, yet

Fig. 91. no colours are produced. If X be a lucid point, ſending rays to the eye at S, through the refracting planes AC and BD of the parallelopipedon; it is plain, that drawing S*t*NX the path of the red ray, and S*p*MX that of the purple ray, theſe rays are equally refracted at both ſurfaces; and therefore the triangles *p*S*t*, MXN are ſimilar; the purple ray, by reaſon of the greater refrangibility on both ſides at *p* and M, deviating more from the direct path than the red; whence they muſt needs cut one another within the priſm, as appears at O, making again the triangles *p*O*t*, MON, ſimilar, or the trapezium S*p*O*t* ſimilar to the trapezium XMON, and therefore they come to the eye, as if they had at firſt proceeded from the ſame point O, and had only been refracted at one of the ſurfaces. And hence it follows, not only that the colours will be generated, but alſo that the angle *p*S*t*, or the apparent breadth of the colours, and other circumſtances, may be determined for any poſition of the eye. And therefore it will be plainer, by making experiments, that objects deeply immerſed in water, will appear a little tinctured with colours, by the refraction of the upper ſurface, to an eye beholding them obliquely. For AC may repreſent the ſurface of the ſtagnant water, and O any object immerſed in it, which the ſpectator at S beholds. And O is eaſily found by drawing the right line SX, cutting the refracting ſurfaces in K and L; and dividing it in O, ſo that SK be to LX as SO to OX, or as KO to OL.

A tranſlation of part of the laſt paper in the optical lectures, concerning the rainbow, p. 285, &c.

There ſtill remains that wonderful ſpectacle of the celeſtial arch, to the explication of which *Cartes* has paved the way; for to him it is owing, that we know that it is formed in the drops of falling rain; which

which is plain from this, that it is never ſeen but Fig.
in rainy weather, where the ſun illuminates the
falling rain; and appears ſometimes in places very
near, as if it was not placed in the air, being fixt
upon the walls of the oppoſite houſes, or rather in
the intermediate ſpace; that water by ſome ſort of
artifice being thrown out in drops ſhows the rain-
bow, and the graſs in a dewy morning, full of
ſmall drops, gives the colours of the rainbow.
To him is alſo owing the moſt ingenious invention
of the refractions of a drop, and the limits thereof,
but the phyſical cauſe he did not ſo happily find
out. That you may underſtand this, ſuppoſe a ray
AN to fall upon the globe NFG at N, and be 92.
thence refracted towards F, where again it is either
refracted towards V, or perhaps reflected to G.
And if it afterwards emerge, it is refracted again
to R, or reflected to H, and ſo on; ſo that ſome
of the rays entering the globe, as NFV preſently
go out, ſuffering no reflection; others as FGR,
after one reflection; and others as GHS after two;
others after three, or even more. Now ſince the
drops of rain, in regard to the diſtance from the
eye of the ſpectator, are exceeding ſmall; they
may be taken for phyſical points; and therefore
there is no occaſion for conſidering their magni-
tudes, but only the angles which the incident rays
make with the emergent ones; for where theſe an-
gles are the greateſt or leaſt, the emergent rays are
each of them denſer; and becauſe for divers ſorts
of rays, there are alſo divers of theſe greateſt or
leaſt angles; each of them thus denſe tending to
different points, will prevail and exhibit their pro-
per colours. Therefore we muſt determine theſe
greateſt or leaſt angles, which the emerging rays
of all ſorts, contain with the incident rays, that
we may rightly perceive the reaſon of theſe phœ-
nomena.

Fig. 92. It is ſhewn in Cor. 1. and 2. Prop. 35. (Part. 1.) that the emergent ray GR is the leaſt inclined to the incident ray AN, when $3RR : II - RR :: CN^2 : ND^2$. And I : 2R :: ND : NE, putting I to R as the ſine of incidence to the ſine of refraction; and ND and NE being thus found, GR will be given in poſition.

For example, in the moſt refrangible rays, the ſine of incidence is to the ſine of refraction, or I to R, as 185 to 138, as is found the neareſt in rain water; and it will be, 57132 : 15181 :: (3RR : II — RR ::) $CN^2 : ND^2$. Therefore $DN = \sqrt{\frac{15181}{57132}} \times CN = \frac{5155}{10000} CN$. Whence by the table of ſines, there is given the arch NL 62 deg. 4 min. Further, ſince I : 2R :: ND : NE :: 185 : 276 :: $\frac{5155}{10000}$ CN : NE, NE will be $= \frac{7691}{10000}$ CN. And then by the table of ſines there is given the arch NF 100 deg. 32 min. Then ſubtract twice the arch NF from the ſum of the arches NL, and 180 deg. or a ſemi-circle, and there will remain 41 deg. 0 min. for the inclination of the rays RG and AN, or for the angle AXR; that is, producing AN and NG to meet in X, and this is the angle under which the inner or blue border of this bow ought to appear, or its leaſt ſemi-diameter.

After the ſame manner for the leaſt refrangible rays, putting the ſine of incidence to the ſide of refraction, as 182 to 138, as I meaſured them, there will be found $ND = \frac{5028}{10000} CN$, and $NE = \frac{7533}{10000} CN$; and thence by the table of ſines, NL will be 60 deg. 22 min. And the arch NF 98 deg. 38 min. And therefore the angle AXR 43 deg. 6 minutes; under which the external, or red

red edge of the bow, will appear. Therefore its greateſt ſemi-diameter is 43 deg. 6 min. From which take the leaſt ſemi-diameter 41 deg. 0 min. there remains the breadth of the bow about 2 deg. 6 min. Or rather, by adding the ſun's diameter (which is 31 min.) 2 deg. 37 min. But as the colours at both edges of the bow, are ſo weak as not to be ſeen for the ſplendor of the adjoining clouds, its breadth as to ſenſe, ſcarce exceeds 2 degrees. Fig. 92.

The dimenſions of the exterior bow are determined in a manner no way different. For it is ſhewn in Cor. 1. and 2. Prop. 36. (Part. I.) that the inclination of the emergent and incident rays HS and AN, is the greateſt, when it is $8RR : II - RR :: NC^2 : ND^2$. And $I : 3R :: ND : NE$. Wherefore putting the numbers 185 and 138 for the ſines of the moſt refrangible rays I and R, as before; we ſhall get $ND = \frac{3157}{10000}$ CN, and $NE = \frac{7064}{10000}$ CN, whence, by the table of ſines, the arch NL = 36 deg. 48 min. and the arch NF = 89 deg. 53 min. And therefore the angle AYS = 52 deg. 51 min. which will be the greateſt ſemi-diameter of this bow. And likewiſe for the ſines of the leaſt refrangible rays, I and R, putting the numbers 183 and 138, there comes out $ND = \frac{3079}{10000}$ CN, and $NE = \frac{6965}{10000}$ CN. Whence by the table of ſines, we ſhall get the arch NL = 35 deg. 52 min. And the arch NF = 88 deg. 18 min. And therefore the angle AYS will be 49 deg. 2 min. the leaſt ſemi-diameter of the bow. Therefore if from the greateſt ſemi-diameter 52 deg. 51 min. there be ſubtracted the leaſt 49 deg. 2 min. and to the remainder be added the ſun's diameter, 31 min. The breadth of this bow will come

Fig. 92. come out 4 deg. 20 min. But by reaſon of the greater obſcurity of this than of the interior bow, I imagine the colours will ſcarce appear further than the breadth of three, or three and a half degrees.

That I may diſtinctly repreſent to view the reaſon of theſe bows, let E, F, G, be the drops any way thrown into the air. SE, SF, SG, the parallel rays of the ſun falling on theſe drops; EM, EN, EO, the different refrangible rays, coming out of the drop E, after one reflection; and FN, FO, FP, and alſo GO, GP, GQ, like rays coming out of the drops F and G; that is, EO, FP, GQ, the moſt refrangible rays, and EM, FN, GO, the leaſt refrangible, &c. Now if the eye of the ſpectator be placed at O, it is plain from the hypotheſis, that among the rays which the drop E emits after one reflection, only the moſt refrangible or blue, as EO fall upon the eye; the reſt as EN and EM, paſſing by from the leſſer refraction; and therefore the blue colour is ſeen at E. But among the rays which G emits after one reflection, the moſt refrangible, as GQ, paſs by the eye, becauſe they are parallel to the ray EO; and the rays of another ſort, ſuppoſe the leaſt refrangible, or red rays as GO, fall upon it; whence the red appears at G; and by the like argument the drop F, in the middle between E and G, ſends the mean refrangible rays, as FO to the eye; the reſt as FN, FP, paſſing by on each ſide; and therefore the green is ſeen at F. The reaſon is the ſame for all drops placed at the ſame apparent diſtances from the axis OR, which paſſes through the ſun and the eye; and therefore the colours will every where appear at theſe diſtances. That is, this coloured bow, whoſe inner ſide is blue, and outer ſide red, and the middle parts of the middle colours, will then appear, the angle OGQ or GOE remaining; and the breadth of this bow will be about two degrees,

grees, as I have ſhewn before; and the like rea- Fig.
ſoning holds good concerning the outer bow; ex- 92.
cept that the order of the colours is contrary, by reaſon of the contrary inflection of the rays. But the drops which are ſituated out of theſe bows on any ſide, can throw no rays at all upon the eye, after one or two reflections, and two refractions; for on the other ſide, they are all mixt together, and become inſenſible; and therefore can exhibit no phœnomena of that kind; but the ſky in theſe places will appear as uſual.

Beſides the phœnomena of colours of which we have been treating, there are ſtill many other things (eſpecially about the colours of very thin tranſparent plates, ſuch as bubbles of water, and air compreſſed between two planes, and the very thin ſkins of ſeveral things), the cauſe and meaſure of which can hardly be accurately determined without mathematical reaſoning; but in theſe things I ſeem to have enlarged too much, and therefore I now betake myſelf to the more abſtracted parts of the methematics. The end.

ART.

Fig.

ART. XVIII.

A Collection of curious, useful, and important Problems.

PROB. I.

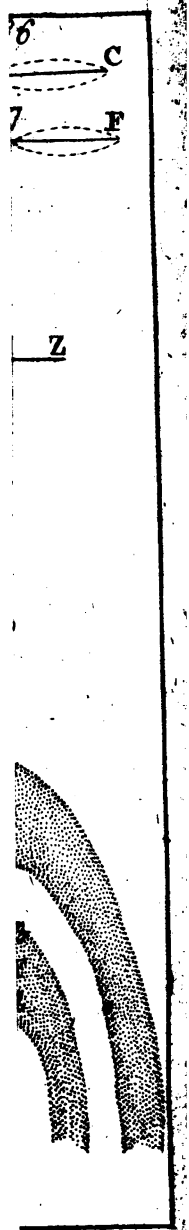
Given an equation of x and y, as $x^a y^b + x^c y^d + x^e y^f + x^g y^h = 0$; to find how many ways it may be transformed; that there may always be one term without x, and another without y.

1. DIVIDE all the terms by x^a, then the equation is $y^b + x^{c-a} y^d + x^{e-a} y^f + x^{g-a} y^h = 0$, which is one way where one term is without x. In like manner, divide all by x^c, and the second term will be without x; which is a second way. Again divide all by x^e, and the third term will be without x. Also divide all by x^g, and the fourth term will be without x. Hence the equation admits of as many different forms as there are terms, wherein there are always one term without x.

2. Again, in the equation $y^b + x^{c-a} y^d + x^{e-a} y^f + x^{g-a} y^h$, setting aside the first term y^b, the equation $x^{c-a} y^d + x^{e-a} y^f + x^{g-a} y^h$ will admit of as many forms, where one term wants y, as there are terms left, as appears by proceeding as in Art. I. Therefore if t be the number of terms in the given equation; the whole number of forms, where one term is without x, and another without y, is $= t \times \overline{t-1}$.

SCHOL.

This supposes that all the indices are different from one another; but if several indices be the same,


6
C
7
F
Z

ſame, the number of transformations will be leſs. Therefore in ſuch caſes, the number will never amount to $tt - t$. Fig.

PROB. II.

A and B *have each a number of ſhillings; and* A*'s number is to* B*'s, in the ratio of* 17.501501501 *&c. ad infinitum, to* 31.917171717 *&c. ad infinitum; and they are the leaſt in that ratio: what are the numbers?*

By Prop. LVI. Chap. III. B. II. *Arithmetic*, the circulating part 501 &c. = vulgar fraction $\frac{501}{999}$. Alſo the circulating part 17 &c. = $\frac{17}{99}$. Therefore 17.501501 &c. = $17\ \frac{501}{999}$; and 31.917 &c. = $31.9\ \frac{17}{99}$. Therefore A's number is to B's, as $17\ \frac{501}{999}$ to $31.9\ \frac{17}{99}$, or as $153 + \frac{501}{111}$ to $287.1 + \frac{17}{11}$, or as $\frac{17484}{111}$ to $\frac{3175.1}{11}$; that is, as 1923240 to 3507378, which in the leaſt numbers will be as 320540 to 584563.

PROB. III.

A uſurer lends out 2000*l. which the borrower is to clear off by quarterly payments, viz.* 1*l. at the end of the firſt quarter,* 2*l. at the end of the ſecond,* 3*l. at the end of the third, and ſo on. In what time will the debt be cleared, reckoning* 5 *per cent. comp. intereſt for all.*

Let p = 2000 the principal, r = 1*l.* and its intereſt at a quarter's end, e = intereſt alone, x = time

Fig. time required; then the money due at the end of the ſeveral quarters will be as follows.

quarters expired.	money then due.	by ſubſtitution.	by reſtitution.
0	p		
1	$rA-1$	$= A$	$= rp - 1$
2	$rB-2$	$= B$	$= rrp - r - 2$
3	$rB-3$	$= C$	$= r^3p - r^2 - 2r - 3$
4	$rC-4$	$= D$	$= r^4p - r^3 - 2r^2 - 3r - 4$
5	$rD-5$	$= E$	$= r^5p - r^4 - 2r^3 - 3r^2 - 4r - 5$
	&c.	&c.	&c.
x	$rG-x$	$= H$	$= r^xp - r^{x-1} - 2r^{x-2} - 3r^{x-3} - 4r^{x-4} \ldots\ldots x$

Then *per* queſt. H or $r^xp - r^{x-1} - 2r^{x-2} - 3r^{x-3}$ &c. $= 0$,

that is, $r^x \times : p - \frac{1}{r} - \frac{2}{rr} - \frac{3}{r^3} - \frac{4}{r^4}$ &c. to x terms $= 0$.

But by Prop. 30. Increments (Schol.) the ſum of x terms of the ſeries $\frac{1}{r} + \frac{2}{rr} + \frac{3}{r^3}$ &c. $= \frac{r}{\overline{r-1}^2} - \frac{1}{r^x} \times \frac{x}{r-1} + \frac{r}{\overline{r-1}^2} = \frac{r}{ee} - \frac{ex+r}{eer^x}$. Therefore $pr^x - r^x \times \frac{1}{r} + \frac{2}{rr} + \frac{3}{r^3}$ &c. $= pr^x - \frac{r^{x+1}}{ee} + \frac{ex+r}{ee} = 0$, and $peer^x - rr^x + ex + r = 0$, reduced $r^x = \frac{r+ex}{r-pee}$; and $x \times \log. r = \log. \frac{r+ex}{r-pee}$, and $x = \frac{\log. \frac{r+ex}{r-pee}}{\log. r} = \frac{\log. 1.42119 + .017233x}{.005395}$; which is eaſily found by

by trial and error. Here $r = 1.012273$, and $e =$ Fig. $.01227$, and x will come out 85.3; ſo that after $85\frac{1}{4}$ quarters the debt will be paid, that is in $21\frac{1}{4}$ years.

SCHOL.

If the queſtion had been to find when the debt is the greateſt poſſible; then the two ſucceeding quarters H and K are equal; that is, $H = K = rH - \overline{x+1}$, whence $\frac{x+1}{e} = H = r^x \times p - \frac{1}{r} - \frac{2}{rr} - \frac{3}{r^3}$ &c. to x terms $= pr^x - \frac{rr^x}{ee} + \frac{ex+r}{ee}$. And $ex + e = peer^x - rr^x + ex + r$, and $\overline{r - pee} \times r^x = r - e$, whence $r^x = \frac{r-e}{r-pee} = \frac{1}{r-pee}$; and $x = \frac{-\log. \overline{r-pee}}{\log. r} = 27.8$; therefore in near 28 months, or 7 years, the debt is the greateſt.

PROB. IV.

A lends B 2000*l. at* 5 *per cent. ſimple intereſt, and is to pay it back, thus,* 1*l. at* 1 *year's end,* 2*l. at* 2 *years end,* 3*l. at* 3 *years end, &c. To find when the debt will be greateſt, when it will be* 85*l.* 15*s. and when all paid.* (Lad. Diary, 1769.)

Put $x =$ time, $r =$ intereſt of $1l. = .05$; then at the end of x years, he would owe $2000 + 2000 \times rx$, if he paid nothing. But ſince he pays 1, 2, 3, &c. pounds in x years, that with its intereſt muſt be deducted; which is $1 + r \times \overline{x-1} + 2 + \frac{2r \times}{x-2}$

Fig. $\overline{x-2}+3+3r\times\overline{x-3}$ &c. to x terms. But by arith. progreſſion,

$$1+2+3\ldots x=\frac{x+1}{2}x.\quad\text{And}$$

$$1+2+3\ldots\text{to }\overline{x-1}:\times rx=\frac{x\times\overline{x-1}}{2}rx,\text{ and}$$

$$1+4+9\ldots\overline{x-1}^2:\times r=\overline{x-1}\times\frac{x}{2}\times\frac{2x-1}{3}=\frac{2xx-3x+1}{6}rx,$$

by Exam. 7. Prop. 10. diff. method.

Therefore at the end of x years he owes

$$2000+2000rx-\frac{x+1}{2}x-\frac{xx-x}{2}rx+\frac{2xx-3x+1}{6}rx;$$

that is, putting $s=2000$, and reducing $s+srx-\frac{rx^3+3xx+\overline{3-r}.x}{6}=$ the debt at x years end. Therefore *per* queſt.

1. $s+srx-\frac{rx^3+3xx+\overline{3-r}.x}{6}=$ max. In Fluxions,

$$srx-\frac{rx^2\dot{x}}{2}-x\dot{x}+\frac{3-r}{6}\dot{x}=0,\text{ whence }r=46\text{ very near.}$$

2. $s+srx-\frac{rx^3+3xx+\overline{3-r}.x}{6}=\pm 85l.\frac{3}{4}$. And the roots extracted $x=$ ſomething leſs than 94, in the firſt caſe; and ſomething more than 94 years, in the ſecond.

3. $s+srx-\frac{rx^3+3xx+\overline{3-r}.x}{6}=0$, and extracting the root, $x=94$ years very near, when all the debt is paid.

PROB.

Fig.

PROB. V.

To find the value of $\sqrt{a+\sqrt{b+\sqrt{a+\sqrt{a}}}}$ *&c. ad infinitum.*

Let $x =$ that value, then $xx = a + \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}$ &c. and $xx - a = \sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a}}}}$ &c. $= x$ the firſt value, becauſe it reverts to the ſame quantity; then $xx - x = a$ and $x^2 - x + \frac{1}{4} = a + \frac{1}{4}$, and $x - \frac{1}{2} = \sqrt{a+\frac{1}{4}}$, and $x = \frac{1}{2} + \sqrt{a+\frac{1}{4}}$.

PROB. VI.

To find the value of $\sqrt{a\times\sqrt{b\times\sqrt{a\times\sqrt{b}}}}$ *&c. ad infinitum.*

Let $x =$ its value, then $xx = a \times \sqrt{b\times\sqrt{a\times\sqrt{b}}}$ &c. and $\frac{xx}{a} = \sqrt{b\times\sqrt{a\times\sqrt{b}}}$, and $\frac{x^4}{aab} = \sqrt{a\times\sqrt{b\times\sqrt{a\times\sqrt{b}}}}$ &c. $= x$, whence $\frac{x^4}{aab} = x$, and $x^3 = aab$, and $x = \sqrt[3]{aab}$.

SCHOL.

Theſe two Problems and their ſolutions are *J. Bernoulli's*; but the ſolutions are certainly wrong, for both the quantities are infinite.

PROB. VII.

To find the ſpecific gravity of a fluid.

Take two bodies A, B, of unequal ſpecific gra-

Fig. vities, but both greater than that of the fluid; make them of equal weights in vacuo. Suſpend them by threads to a pair of ſcales as uſual, and immerſe them both in the fluid, and get the difference of the weights. Then this difference of weight = weight of a quantity of the fluid, equal to the difference of the magnitudes of the bodies A, B.

For let A be the bulk of that ſpecifically heavier, B that of the lighter; x = what A loſes in the fluid, y = what B loſes. And when they are immerſed, (put w = weight of A or B, d = diff. weights in the fluid.)

$w - x$ = A's weight in the fluid.

$w - y$ = B's weight in the fluid.

Then ſince B is bigger, it loſes more than A, and A is heavier in the fluid; whence $w - x - \overline{w - y}$, or $y - x$, or d = difference of the weights of the bulks of water A and B; that is, d = a bulk of water = B — A.

Or thus.

If two ſuch bodies be made in equilibrio in the fluid; and weighing them in vacuo; the difference of the weights will be equal to the weight of a quantity of the fluid, equal to the difference of the bulks of the two bodies.

Note, the more unequal the ſpecific gravities of A and B are, the more exactly that of the fluid will be found.

PROB. VIII.

Two known metals being mixt together, as gold and copper, to find the quantity of each.

Let magnitude of the mixture = A, ſp. grav. = a

of the copper = B, ſp. grav. = b

of the gold = A — B, ſp. grav. = g

Then

Fig.

Then the weights of

the gold	$= gA - gB$
copper or allay	$= bB$
mixture	$= aA$

Then $gA - gB + bB = aA$, reduced $B = \frac{g-a}{g-b} A$.

Here the magnitude is known, by the weight of as much water, which is = absolute weight — weight in water.

Cor. *Weight* A : *weight* B : : (Aa : Bb : :) $a \times g - b : b \times g - a$.

Or thus.

Let x = weight of gold,
y = weight of copper,
a = weight of the compound,
b = weight of the compound in water,
c = specific gravity of gold,
d = specific gravity of copper,

then $c : 1 : : x : \frac{x}{c}$ = weight of water of the bulk of x.

$d : 1 : : y : \frac{y}{d}$ = weight of water of the bulk of y.

then $x - \frac{x}{c}$ = weight in water of the gold,

$y - \frac{y}{d}$ = weight in water of the copper.

whence $x - \frac{x}{c} + y - \frac{y}{d} = b$, or $\frac{c-1.x}{c} + \frac{d-1}{d} y = b$.

therefore $c - 1.dx + d - 1.cy = bcd$. but $y = a - x$.

Fig. therefore $\overline{c-1}.dx + \overline{d-1}.c \times \overline{a-x} = bcd$,

reduced $x = \frac{bcd - acd + ac}{c-d} = \frac{a-d \times \overline{a-b}}{c-d} c$.

and $y = \frac{c.\overline{a-b} - a}{c-d} d$.

Cor. *Let* $g =$ *ſpecific gravity of the mixture, then* $g = \frac{a}{a-b}$, *and* $a - b = \frac{a}{g}$, *therefore* $x = \frac{g-d}{c-d} \times \frac{ac}{g}$, *and* $y = \frac{c-g}{c-d} \times \frac{ad}{g}$. *Whence* $\overline{c-d}.g$:

$$a :: \begin{cases} \overline{g-d}.c : x. \\ \overline{c-g}.d : y. \end{cases}$$

PROB. IX.

If a piece of ground is to be planted with trees, ſo that they may be at a certain diſtance from one another. To find the order they muſt be planted in, to have the moſt trees poſſible.

Since there are only 3 regular figures which can fill a ſpace, viz. equilateral triangles, ſquares, and hexagons; therefore if the trees be planted in any regular order, it muſt be in the centers of ſome of theſe. Now let $r =$ radius of a circle wherein a tree ſtands, and ſpreads itſelf to that diſtance every way. Now whatever figure a tree ſtands in, the inſcribed circle muſt be the ſame in all of them, whether a triangle, ſquare or hexagon. Therefore we muſt enquire what proportion each of theſe figures have to the inſcribed circle.

An equilateral triangle being inſcribed in a circle, the ſide is $= r\sqrt{3}$ (by Prop. XLI. B. IV. Geometry), and (by Cor. 2.) the perpendicular $= \frac{1}{2}r$, and the ſides being 3, the area $= 3 \times \frac{1}{4}rr\sqrt{3} = \frac{3}{4}rr\sqrt{3}$. And ſince the perp. to the ſide of the circum-

circumſcribing triangle is twice as much being $= r$, Fig. therefore the area of the circumſcribing circle is 4 times as great $= 3rr\sqrt{3}$; and this is to contain one tree.

The area of the circumſcribing ſquare, it is evident, is $= 4rr$, which alſo contains a tree.

In the hexagon, the ſide of the inſcribed figure being r, the perpendicular is $\sqrt{rr - \frac{1}{4}rr} = \frac{r}{2}\sqrt{3}$, and its area $= 6 \times \frac{rr}{4}\sqrt{3} = \frac{3rr}{2}\sqrt{3}$. But the perpendicular on a ſide of the circumſcribing hexagon being r; it will be as $\frac{1}{4}rr : rr :: $ area inſcribed $\frac{3rr}{2}\sqrt{3} : 2rr\sqrt{3}$, for the area of the circumſcribing hexagon, which likewiſe contains a tree. Hence the areas of the triangle, ſquare, and hexagon, (that each may contain a tree) are as $3\sqrt{3}$, 4, and $2\sqrt{3}$; and theſe are as the numbers 3, $2\frac{1}{3}$ and 2 nearly. Hence the hexagons are moſt advantageous, as taking up leſs ground, the triangles leaſt proper.

Hence for every 3 trees planted in hexagons, there will but be 2 planted in triangles, in a given quantity of ground. And for every 7 planted in hexagons, there will but be about 6 planted in ſquares, in the ſame ground.

Cor. By the above reaſoning it appears, that if the trees be placed at the angular points of the figures, the advantage lies the contrary way; triangles the beſt, ſquares the next, and hexagons the leaſt; and in proportion as above. For a hexagon is compoſed of equilateral triangles, whoſe angular points meet in the center of it. And the angular points of hexagons may be conſidered as the centers of triangles; the angular points of which triangles meet in the centers of the hexagons.

Fig. And perhaps this is an eaſier way in the practice of planting.

The ſame rules may be applied to any other things, which are to be placed at equal diſtances.

PROB. X.

To place a number of ſmall ſpherical bodies, in the leaſt ſolid ſpace.

94. 1. If ſeveral of theſe ſpheres be ſo placed, that all their centers may be in the ſame plane, and all touch one another; then if the centers of any three touching ſpheres, be joined by right lines, they will form an equilateral triangle as *abc*. Now if upon this bed, or layer of bodies, another like layer be placed, ſo that the ſpheres may fall into the alternate ſpaces of the former layer, and ſo on for more layers; then it is evident, that every 4 ſpheres that touch one another, will have their centers at the 4 angles of a tetrahedron; and this is the conſtitution of a ſyſtem of ſpheres, diſpoſed after this manner.

95. 2. If the ſpheres are ſo diſpoſed in ſquares, ſo that any 4 touching ſpheres, having their centers joined by right lines, they will form a ſquare *abdc*. And if upon this layer or bed another ſuch layer be placed, ſo that the ſpheres may lie in the hollows or ſpaces of the former, and ſo on for other layers. Then it will appear, that any 6 ſpheres touching one another (and lying in 3 beds or layers) will have their centers at the 6 angles of the octaedron; or, which is the ſame thing, any 5 touching ſpheres (in 2 layers) will have their centers in the 5 angles of a half octaedron. And this is the conſtitution of a ſyſtem of ſpheres, diſpoſed after this manner.

3. To find the ſolid ſpace taken up by any number of theſe ſpheres, we muſt firſt find the area of the

the baſe for containing two rows. When the length of the ſpace is given, the number of ſpheres in any row is the ſame in both ſyſtems; but the breadth of two rows will differ in the two ſyſtems. The breadth of the ſpace in the firſt ſyſtem = height of the equilateral triangle $= r\sqrt{3}$, (r being the radius of a ſphere); and the breadth of two rows in the ſecond ſyſtem $= 2r$, as appears by the figures; this is reckoning from the centers; but reckoning from the outſides of two rows, the breadth $= 2r + r\sqrt{3}$ in the firſt ſyſtem, and breadth $= 4r$ in the ſecond ſyſtem. Fig. 94. 95.

Again for the height, the diſtance of the center of a ſphere, in a row of the ſecond layer = height of the regular pyramid $= 2r\sqrt{\frac{2}{3}}$ in the firſt ſyſtem; and the diſtance in the ſecond ſyſtem = height of half the octaedron $= r\sqrt{2}$; but this is alſo reckoning from the centers. Therefore reckoning from the outſides, the height of two rows, in the firſt ſyſtem $= 2r + 2r\sqrt{\frac{2}{3}}$, and in the ſecond ſyſtem height $= 2r + r\sqrt{2}$.

4. Hence, taking two rows in each layer; and the 4 rows will be included in a ſpace in form of a parallelopipedon, whoſe length in both ſyſtems is the ſame, but the baſe in the firſt ſyſtem $= r \times \overline{2 + \sqrt{3}} \times r \times \overline{2 + 2\sqrt{\frac{2}{3}}}$, and in the ſecond ſyſtem $= 4r \times r \times \overline{2 + \sqrt{2}}$; therefore the ſpaces to contain the ſame number of ſpheres in the firſt and ſecond ſyſtems, are as $13.557rr$ to $rr \times 13.656$; therefore theſe ſpaces are as 137 to 138; being nearly equal, but the firſt ſyſtem has rather the advantage.

5. In equal ſpaces the denſity by the pyramidal method (Fig. 94.) is to the denſity by the octagonal way (Fig. 95.) is as 138 to 137. For the denſity is reciprocally as the ſpace taken up by a given quantity of matter.

Fig. 94. 95. 6. If the ſpheres are ſo diſpoſed that 8 of them having their centers connected with right lines, theſe lines will form a cube; then the baſe or ſection of ſuch a ſyſtem will be $4r \times 4r$ or $16rr$, which give a ſpace greater than the former, in the ratio of 16 to 13.557 or 13.656, for the ſame number of ſpheres.

7. Denſity of ſolid ſpace : denſity ſyſt. art. 6 : : cube of the diameter : to the ſphere, that is, as $8a^3 : \frac{4}{3}a^3 \times 3.1416$.

And denſ. ſyſtem art. 6 : denſity pyram. ſyſtem : : 13.557 : 16, therefore denſity of ſolid ſpace : denſity of the pyramidal ſyſtem : : $8a^3 \times 13.557 : 16 \times \frac{4}{3}a^3 \times 3.1416 : : \frac{3}{8} \times 13.557 : 3.1416 : :$ 5.084 : 3.1416.

8. If a great heap of ſmall ſpheres be ſhaken together in a box; they will diſpoſe of themſelves, according to one of the two firſt ſyſtems, and moſt probably according to the firſt or pyramidal one, where all their centers are at the angles of regular pyramids.

9. If the ſpheres are ſo diſpoſed, that each ſphere in the upper layer lies between 2 ſpheres in the under one, ſo that all the three centers may be in a perp. plane, and make an equilateral triangle; then the breadth $= 2r$, and height $= r\sqrt{3}$; and whole breadth $= 4r$, and whole height $= 2r + r\sqrt{3}$. Therefore the baſe $= 4r \times \overline{2r + r\sqrt{3}} = 8rr + 4rr\sqrt{3} = rr \times \overline{8 + 6.928} = rr \times 14.928$; which is a greater ſpace than either of the firſt.

10. Hence the denſities in the 4 ſyſtems art. 1, 2, 9, 6; are reſpectively as $\frac{1}{13.557}$, $\frac{1}{13.656}$, $\frac{1}{14.928}$, and $\frac{1}{16}$, and of ſolid ſpace $\frac{1}{\frac{8}{3} \times 3.1416} = \frac{1}{8.3776}$, by Art. 7.

PROB.

Fig. 96.

PROB. XI.

ASB *is a compound pendulum, made of the two bodies* A *and* B, *moving about the center of motion* S, *and vibrating in a ſecond. To place them ſo that the contraction or expanſion of the rods* Sa, Sb, (*made of different metals*) *ſhall cauſe no difference in the time of vibration; if it be poſſible.*

It is plain, the bodies A, B, muſt be on different ſides of S; for if they be on one ſide; when one rod is expanded the other will be expanded, and the pendulum becomes longer, and the time of vibration longer. And the contrary when they both contract.

Let $d =$ diſtance of the center of Oſcillation from S $= 39.2$ $a =$ SA, $b =$ SB.

Then by Mechanics, $\frac{Aaa + Bbb}{Aa - Bb} = d$. Now let a expand to a', and b to b', both increaſing their length; then to find whether $\frac{Aa'a' + Bb'b'}{Aa' - Bb'}$ be ſtill $= d$, put $A = pB$, p being a fixt number; then we have, by the firſt equation, $\frac{pBaa + Bbb}{pBa - Bb} = d$, or $\frac{paa + bb}{pa - b} = d$, now ſince both a and b are increaſed, the whole $paa + bb$ is greater than before, being increaſed as the ſquare of a and b. But $pa - b$ cannot be increaſed in the ſame proportion, either of them being only increaſed as a or b. Let $qa = b$, then $\frac{paa + qqaa}{pa - qa} = d$, $\frac{p + qq}{p - q} a = \frac{Aaa + Bbb}{Aa - Bb}$, and $\frac{p + qq}{p - q} a'$ is greater than d, ſince p and q are fixt numbers. And the ſame thing will

Fig. 96. will follow in contractions, for then $\frac{p+qq}{p-q}a$ will be leſs than d.

Hence it is impoſſible to make a compound pendulum of two bodies, and with two rods of any different metals, ſo as to keep time truly, when they expand or contract.

Otherwiſe thus.

Since $\frac{paa+bb}{pa-b} = d$, let the expanſion of a and b, be x and y; then $\frac{p\times\overline{a+x}^2+\overline{b+y}^2}{p\times\overline{a+x}-b-y} = d$, if it can be done; that is, $\frac{paa+2pax+bb+2by\ \&c.}{pa+px-b-y} = d$, but ſince $\frac{paa+bb}{pa-b} = d$, therefore the remainder $\frac{2pax+2by}{px-y} = d$, let $y = nx$, then $\frac{2pax+2nbx}{px-nx} = \frac{2pa+2nb}{p-n} = d$, or $\frac{paa+bb}{pa-b} = \frac{2pa+2nb}{p-n}$, but no numbers can be found that will make them equal, for the latter will always be greater. For if $b=a$, then will $\frac{p+n}{p-n}\times 2a, = d$, which is greater than $\frac{paa+aa}{pa-a}$, and if b be increaſed n will be increaſed, and p diminiſhed; yet this quantity will always be the greater.

PROB.

Fig.

PROB. XII.

DF *is a wafer barometer, wherein the water ſtands at the height* AO *in the tube at the ſurface of the earth; then ſuppoſe it carried to the top of a hill, where the water riſes to* B; *to find the height of the hill.* 97.

Suppoſe the veſſel DF and pipe BA, to be cylinders; the diameter of the veſſel DF $=$ D, diameter of the pipe $=d$, and $p=$ ED the height of the internal air, $n=$ AO the height of the water in the pipe, $x=$ OB the aſcent of the water, all in inches. And $f=34$ feet the height of a column of water of equal weight with the atmoſphere; then $850f=$ height of a uniform atmoſphere, $y=$ altitude of the hill.

Suppoſe, in going to the top of the hill, it riſes from O to B in the pipe, and falls from E to C in the veſſel. Then the cylinders OB and CF are equal, therefore $EC \times DD = xdd + EC \times dd$, and $EC = \frac{xdd}{DD-dd} = xq$. Now OB or x would be the height of the hill, if air was as heavy as water, therefore $850x =$ its height in air, and $\frac{850x}{1.2} =$ height in air; that is $y = 71x$ in feet. But this is on ſuppoſition, that the preſſure at C is as great as at E, by which it is raiſed through the height x; but the preſſure at C being leſs; the height muſt be increaſed in the ratio of the preſſures at firſt and laſt, that is as DC to DE, and moreover increaſed by the addition of EC, by which the pipe of water is increaſed. Whence the height $=$ $\frac{DC}{DE} + EC \times 71x = \frac{DE+EC}{DE} + EC \times 71x =$

1 +

Fig. 97. $1 + \frac{qx}{p} + qx \times 71x = 1 + \frac{p+1}{p} qx \times 71x$; that is $y = 1 + \frac{p+1}{p} \times \frac{ddx}{DD - dd} \times 71x$, the true height in feet.

Cor. *If* D *is very large, the quantity* qx *may be neglected, and then* $y = 71x$, *very near.*

PROB. XIII.

To find to what diſtance a fowling piece will kill; having the diſtance to which one of a given length will kill.

Let f = force of the powder, s = ſpace deſcribed, v = velocity generated, t = time, b = body moved, or the ſhot. Then (Prob. I. Sect. III. Fluxions,) vv is as $\frac{fs}{b}$. Then,

1. Suppoſe f the force of the powder within the barrel to be uniform; then the velocity will be uniformly accelerated to the mouth of the barrel; and f and b being given, $\frac{vv}{2}$ will be as $\frac{fs}{b}$, and v as $\sqrt{\frac{fs}{b}}$. Therefore put $V = v$, $P = f$, $L = s$, the length of the barrel; and $S = b$ the charge of ſhot, and we have V as $\sqrt{\frac{PL}{S}}$. And therefore P, S, being given, the velocity at the muzzle will be as the ſquare root of the length.

Let x = diſtance the ſhot flies, V = velocity at the gun's mouth, v = killing velocity. Then (Prob. XIII. Sect. III. Fluxions,) $\frac{3px}{8dg} = 2.3025 \log. \frac{V}{v}$, and

$3px$

$\frac{3px}{2.3025\times 8dq} = \log.\frac{V}{v}$; whence $x = \frac{8dq\times 2.3025}{3p}$ Fig. 1 $\log.\frac{V}{v}$, where d is the diam. of a pellet of ſhot, q = denſity of lead, = 10000, p = denſity of air = 1. Therefore $x = 61402d \times \log.\frac{V}{v}$, or $x = 5117d \log.\frac{V}{v}$ in feet, d being inches. Therefore V and v being known x is had.

Since V the velocity at the muzzle is found to be as $\sqrt{\frac{PL}{S}}$; let $V = a\sqrt{\frac{PL}{S}}$; then $x = 5117d \log. a\sqrt{\frac{PL}{S}}$, or $x = 2558d \log. aa \times \frac{PL}{S}$, putting $v = 1$. Then put P, L, S = 1, for a gun that can kill $f(x)$ feet; to be found by experience. Then $f = 5117d \times \log. a$, and a = number of the logarithm $\frac{f}{5117d}$. Then a being known, x will be known for any values of P, L, S.

Note, if one ſort of powder be ſtronger than another, P muſt be increaſed in that ratio. *Note alſo*, the different diameters of the bores of guns, make no difference as to the killing diſtance.

EXAM. 1.

If $f = 90$, $d = \frac{1}{12}$, then $\log. a = .211$, and $a = 1.63$; whence $x = 426 \times \log. 1.63 \times \sqrt{\frac{PL}{S}}$, for any values of P, L, S.

EXAM. 2.

Let $d = \frac{1}{8}$, $f = 90$, then $5117d = 640$, and log.

Fig. log. $a = .1407$, and $a = 1.3825$, and $x = 640$ log. $1.382 \sqrt{\frac{PL}{S}}$.

2. If the force of the powder decreaſes to a certain diſtance, ſo as to be as the diſtance from the extremity. Put $l =$ length of that ſpace or diſtance, $y =$ diſtance from the breach, then the force will be as $l - y$. Then vv is as $\left(\frac{Fs}{b}\right.$ or$\left.\right)$ $\frac{l-y \cdot y}{b} = \frac{ly - yy}{b}$, and v^2 as $\frac{ly - \frac{1}{2}yy}{b}$, whence v is as $\sqrt{\frac{2ly - yy}{b}}$; and when $y = l$, then v is as $\frac{l}{\sqrt{b}}$, when l is the length of the bore; but if $g =$ length of the bore, then the velocity at the muzzle will be $\sqrt{\frac{2lg - gg}{b}}$, or $\sqrt{2lg - gg}$, when b is given.

Having got $V = \sqrt{2lg - gg}$ the velocity at the muzzle, then $x = 5117d \times \log. \frac{V}{v}$ as before, which determines the diſtance to kill, upon this law of force.

3. If the force of the powder within the gun, be reciprocally as the expanſion or ſpace, which at firing is $= c$, the ſpace taken up by the powder; let $y =$ diſtance from the breach as before, then vv is as $\frac{\dot{y}}{y}$, and v^2 as log. y, and corrected v^2 is as log. $\frac{y}{c}$, and v as $\sqrt{\log. \frac{y}{c}}$. Whence the diſtance x will be had as before.

But

But as to the force, which of these laws obtains, is uncertain; and different suppositions, give different conclusions. And this is a matter that must be determined by experiments. But the law being known, every thing else becomes known. Fig.

SCHOL.

To find the strength of a gun barrel in every point. The strength must be as the force, and time of acting. In the first case before laid down, the force is uniform, and therefore the strength is as the time of acting, and that strength is as the thickness of the metal. The time of describing the whole length of the barrel, to the time of describing the length y is as $\sqrt{l}$ to $\sqrt{y}$. Therefore the time of acting on a point at the distance y from the breach is $\sqrt{l} - \sqrt{y}$; therefore the thickness of the metal at the distance y from the breach is as $\sqrt{l} - \sqrt{y}$; upon the supposition of uniform force. And the force will be nearly so, if the powder don't fire all at once, but gradually.

PROB. XIV.

DCFB *are two tubes or cylindric vessels that communicate*; DC *is close at top*, FB *open at top. Water stands in both at the level* CR, *and* DC *contains inclosed air of the same density as the external air. If the tube* BF *be filled with water as high as* B, *to find the space* BO *it will descend in* BR. 98.

When the quantity of water BR is put into the tube BR, it will rise to E in the tube DC. To find BO or EC. Let BO $= x$, EC $= y$, EA $= D$, AF $= d$. Then the cylinder QO $=$ cylinder CE. Whence $DDy = ddx$, and $y = \frac{ddx}{DD} = qx$. Let the horizontal

Fig. 98. horizontal line EF be drawn. Let f = height of water 34 feet, the weight of the atmosphere, p = DC, QR = b.

Then the preſſure at EA = preſſure at AF, to preſerve an equilibrium. The preſſure on EA = $\frac{DC}{DE} f$, by the compreſſing force of the air. The preſſure on AF = $f + AO$; therefore $\frac{DC}{DE} f$, or $\frac{DE + EC}{DE} f = f + AO$, or $f + \frac{EC}{DE} f = f + AO$, and $\frac{EC}{DE} f = AO$, and $EC \times f = DE \times AO$, or $yf = \overline{p - y} \times \overline{b - x - y}$, and $\frac{fy}{p - y} = b - x - y$, and $x = b - y - \frac{fy}{p - y}$, put $q = \frac{dd}{DD}$, $r = 1 + q$; then the equation reduced is $\overline{fq + bq + pr}.x - qrxx = pb$, and $y = qx$.

Cor. *In ſmall heights,* $x = \frac{pb}{fq}$, *nearly.*

PROB. XV.

If a common man 2 yards high can juſt bear 26lb. at arm's end, and the weight of his arm, there be 4lb. to find the ſize of that man that can bear the greateſt weight poſſible at arm's end.

Let d = whole weight at arm's end = 30*lb.* and the arm being = half the length of the body = a, the body = $2a$, x = length of the ſtrong man's arm, y = the additional weight it can bear at arm's end, b = 4*lb.*

The

The weights of ſimilar ſolids, are as the cubes of the lengths, therefore $a^3 : x^3 :: b : \frac{bx^3}{a^3}$ = weight of the ſtrong man's arm, at arm's end; to which add the additional weight y, which is to be a maximum; then $\frac{bx^3}{a^3} + y$ = total weight at arm's end. Fig. 98.

The ſtreſs on the muſcles at the ſhoulder, is as the weight and length of the arm (or leaver); divided by the diſtance of the fulcrum; therefore the ſtreſs is as the weight; becauſe the length of the arm, and diſtance of the fulcrum, is in a given ratio.

And the ſtrength of the muſcles is but as the ſquare of the diameter; or as aa and xx. But the ſtreſs is as the ſtrength, therefore $aa : xx :: d : \frac{bx^3}{a^3} + y$, for the common man, and the ſtrong man; therefore $\frac{bx^3}{a} + aay = dxx$, and $aay = dxx - \frac{bx^3}{a}$ = max. Whence $2dx\dot{x} - \frac{3bx^2\dot{x}}{a} = 0$, and $3bx = 2ad$, and $x = \frac{2ad}{3b} = \frac{2 \times 30}{3 \times 4} a = \frac{60a}{12} = 5a$; therefore the ſtrongeſt man will be 5 times higher than the common man.

SCHOL.

In Prob. 187, Algebra, I have computed the height of a man that can lift nothing, to be $\frac{d}{b} a = 7\frac{1}{2}a$. Therefore the height of the common man, the ſtrong man, and the great man, will be a, $5a$, and $7\frac{1}{2}a$.

Fig.

PROB. XVI.

Two animals of the ſame kind being given, a greater and a leſſer; to find the proportion of the velocities they can move with, in jumping, running, &c.

Let M, m, be the matter or weight of the two bodies; L, l, their lengths or heights; V, v, the velocities of motion.

By Mechanics (ſee Prob. I. Sect. III. Fluxions.) $\dot{v}v$ is as $\frac{fs}{b}$, and vv as $\frac{fs}{b}$. The force (f or) ſtrength of the muſcles, is as the ſquares of their diameters, or as LL to ll. And (s or) the contraction of the muſcles, is as their lengths, or as L to l; and (b or) the weight of the bodies are as M to m, that is as L^3 to l^3. Therefore $V^2 : v^2 :: \frac{LL \times L}{L^3} : \frac{ll \times l}{l^3}$, that is as 1 to 1; therefore $V = v$. And the velocities generated are the ſame in both, by this law of motion, expreſſed by $vv \propto \frac{fs}{b}$. Therefore the different ſize of bodies make no difference in the degree of motion they are capable of performing.

PROB. XVII.

If two weights a and b, be ſuſpended at a rope going over a pulley. To find the ſpace through which the greater body a deſcends, or the leſſer b aſcends, in a ſecond of time.

By Schol. Prop. VI. Mechanics, we find $s \propto \frac{Ftt}{b}$, for uniformly accelerated motion. Let $b =$

16.1 the feet deſcribed in a ſecond by one body falling freely, as a. F = weight of that body $= a$, $s = b$. Alſo in the two bodies x = ſpace deſcribed in a ſecond, F = $a - b$, and $b = a + b$, therefore we have $b : \frac{att}{a} :: x : \frac{a-b}{a+b}\,tt$, whence $x = \frac{a-b}{a+b}\,b$, t being the ſame in both. Fig. 99.

EXAM.

If $a = 2b$, then $x = \frac{2b-b}{2b+b}\,b = \frac{1}{3}b$.

If $a = 3b$, then $x = \frac{3b-b}{3b+b}\,b = \frac{2}{4}b = \frac{1}{2}b$.

If $na = mb$, then $x = \frac{m-n}{m+n}\,b$.

SCHOL.

If it was required to find the time of deſcribing any given ſpace S. Having found the ſpace x deſcribed in 1 ſecond, it will be $x : 1^2 :: S :$ ſquare of the time $= \frac{S}{x}$, and the time $= \sqrt{\frac{S}{x}} = \sqrt{\frac{S.\overline{a+b}}{b.a-b}}$.

PROB. XVIII.

In the fire engine, given the weight or ſtrength of the atmoſphere upon the piſton, and the length of the ſtroke; to find the water to be drawn at a ſtroke; ſo that the greateſt quantity ſhall be drawn in a given time; ſuppoſing the force uniform. 100.

Let a = weight of the atmoſphere, y = weight of the water, AB = BC, t = time of C's deſcending to gain a new ſtroke.

Then the force by which the water is drawn up is $a - y$, and the weight $= y$; and (by Schol.

Fig. 100. Prop. VI. Mechanics), the time of raiſing y thro' a given height will be $\sqrt{\frac{\text{body}}{\text{force}}}$ $\left(t \propto \sqrt{\frac{bs}{p}}\right)$, that is as $\sqrt{\frac{y}{a-y}}$; then $t + \sqrt{\frac{y}{a-y}}$ = time of raiſing y, or the whole time of making a ſtroke. Then to find how much will be drawn up in the given time 1, $t + \sqrt{\frac{y}{a-y}}$ (time) : y (weight) : : 1 (time) : $\frac{y}{t+\sqrt{\frac{y}{a-y}}}$ the weight required = max. put $x = \sqrt{\frac{y}{a-y}}$, then $xx = \frac{y}{a-y}$, and $2x\dot{x} = \frac{a\dot{y}}{\overline{a-y}^2}$, and $\dot{x} = \frac{a\dot{y}}{\overline{a-y}^2 \times 2\sqrt{\frac{y}{a-y}}}$. Whence $\frac{y}{t+x} = m$, and $\frac{t\dot{y} + x\dot{y} - \dot{x}y}{\overline{t+x}^2} = 0$, and $t\dot{y} + x\dot{y} - y\dot{x} = 0$, that is, $t\dot{y} + \dot{y}\sqrt{\frac{y}{a-y}} = y\dot{x} = \frac{ay\dot{y}}{\overline{a-y}^2 \times 2\sqrt{\frac{y}{a-y}}} = \frac{ay\dot{y}}{2\sqrt{y \times \overline{a-y}^3}}$, whence $2t\sqrt{y \times \overline{a-y}^3} + 2y \times \overline{a-y} = ay$, and $2t\sqrt{y \times \overline{a-y}^3} = 2yy - ay$, and $4tt \times \overline{a-y}^3 = \overline{2y-a}^2 \times y$; and the root of this cubic equation will be the value of y for the weight.

If we conſider only the time of aſcent, then t is ſet aſide, and then $y\sqrt{\frac{a-y}{y}}$ = max. and $\frac{ay^2 - y^3}{y}$ or $ay - yy = m^2$, and $a\dot{y} - 2y\dot{y} = 0$, and $a = 2y$, or $y = \frac{1}{2}a$.

PROB.

Fig.

PROB. XIX.

A pendulous body or door BE, *whoſe weight is given; being ſhot at with a bullet, whoſe weight is given, and the arch the door deſcribes after the ſtroke, being alſo given; to find the velocity of the bullet.* 101.

Let weight of the pendulum BD $= w = 56\frac{3}{16} lb.$
weight of the bullet $= m = \frac{1}{12} lb.$
the cord of the arch deſcribed by E $= p =$ $17\frac{1}{4}$ inches.
diſt. center of gravity BD $= b = 52$ inch.
diſt. center of gyration BR $= r = 57.08 = \sqrt{bc}$.
diſt. center of oſcillation BA $= c = 62\frac{2}{3}$.
diſt. center of the pendulum BC $= d = 66$.
velocity bullet impinging at C $= = x$.
BE $= g = 71\frac{1}{3}$, $f = 16.1$ feet.

$g : p :: c : \frac{cp}{g}$ = cord of the arch deſcribed by the center of oſcillation A, and $\frac{cpp}{17 \times 2gg}$ = verſed ſine of that arch in feet; and $\frac{p}{g}\sqrt{\frac{2fc}{12}}$ = velocity of A, by falling through that arch $= v$.

By the motion of the pendulum, $\frac{rv}{c}$ = velocity of the point R. And (Cor. 5. Prop. 59. Mechanics), $\frac{wrv}{c}$ = momentum of the pendulum at R. And mx = momentum of the bullet againſt C, where it impinges. And (Cor. 5. *ib.*) $\frac{dmx}{r}$ = force of the bullet againſt R, to give the ſame motion to the pendulum; therefore $\frac{dmx}{r} = \frac{wrv}{c}$, and

Fig. 101. $x = \frac{wrrv}{dmc} = \frac{wbcv}{dmc} = \frac{wbv}{dm} = \frac{pwb}{dmg}\sqrt{\frac{2fc}{12}} =$ 1672 feet in a second.

Note, I suppose the bullet to be at rest after the stroke, tho' it really moves along with the door or pendulum. But this makes but a trifling difference.

SCHOL.

To find the weight of a pendulum placed in C, *on which the bullet impinging, shall cause it to describe the same angle about* B.

Let Z = weight sought, then $c : d :: v : v \times \frac{d}{c}$ = velocity of the body Z, at the point C.

And the momentum of Z at C, is $= Zv \times \frac{d}{c} = mx = \frac{wbv}{d}$ the momentum of the bullet. Therefore $Z = \frac{wbc}{dd} = 42\frac{1}{12}lb.$

PROB. XX.

102. *To find the center of gravity of the logarithmic space* ABC; AB *being parallel to the asymptote* FE.

Let GF = 1, AD = a, FD = b, DE or AB = x, BC = y, t = subtangent; then by nature of the curve, $\overline{a+y} \times \dot{x} = t\dot{y}$, and $y\dot{x} = t\dot{y} - a\dot{x}$, and fl. $y\dot{x} = ty - ax$ = area ABC.

But $yx\dot{x}$ = fluxion of the momentum of ABC $= txy - ax\dot{x}$, and fl. $yx\dot{x}$ or fl. $txy - ax\dot{x} = txy - p - \frac{ax^2}{2}$. This in fluxions is $tx\dot{y} + ty\dot{x} - \dot{p} - ax\dot{x} = tx\dot{y} - ax\dot{x}$, therefore $\dot{p} = ty\dot{x} = tt\dot{y} - ta\dot{x}$, whence $p = tty - tax$. Therefore fl. $yx\dot{x} = txy - tty$

Fig. 102.

$ty + tax - \frac{ax^2}{2}$ = momentum of ABC. Whence d, the distance of the center of gravity from A, is $= \frac{txy - tty + tax - \frac{1}{2}ax^2}{ty - ax}$, or $d = \frac{ty \times \overline{x - t} + \overline{t - \frac{1}{2}x} \times ax}{ty - ax}$.

Cor. 1. *When* $a = 1$, $d = \frac{ty \times \overline{x - t} + \overline{t - \frac{1}{2}x} \times x}{ty - x}$.

Cor. 2. *The distance of the center of gravity from* BC, *is* $= t - \frac{\frac{1}{2}axx}{ty - ax}$.

For $x - d = x - \frac{ty \times \overline{x - t} + \overline{t - \frac{1}{2}x} \times ax}{ty - ax} = \frac{tty - tax - \frac{1}{2}axx}{ty - ax} = t - \frac{\frac{1}{2}axx}{ty - ax}$.

Again, to find the distance from the axis of suspension AB. The fluxion of the momentum $= \frac{1}{2}y \times y\dot{x} = \frac{1}{2}y \times \overline{t\dot{y} - a\dot{x}} = \frac{1}{2}tyy - \frac{1}{2}ayx = \frac{1}{2}tyy - \frac{1}{2}a \times \overline{t\dot{y} - a\dot{x}} = \frac{1}{2}tyy - \frac{1}{2}aty + \frac{1}{2}aax$. And the fluent $= \frac{ty^2}{4} - \frac{aty}{2} + \frac{aax}{2}$, the momentum; therefore $\frac{ty^2 - 2aty + 2aax}{4 \times ty - ax}$ = distance of the center of gravity from AB.

PROB. XXI.

To find the nature of the curve ADE, *which lying with its plane parallel to the horizon, and supposed to be of a given thickness* q; *and being fixt with its end in the wall* ER; *it shall be equally strong every where to support its own weight.* 103.

The meaning of the question is this, let C be the

Fig. 103. center of gravity of the area ABD, in the axis AB. Then the area BAD × BC = BD × qq.

Let AB = x, BD = y, then area BAD = fl. $y\dot{x}$. And by the nature of the center of gravity, AC = $\frac{\text{fl. } y\dot{x}\dot{x}}{\text{fl. } y\dot{x}}$, the point of ſuſpenſion being at A. Therefore BC = $x - \frac{\text{fl. } y\dot{x}\dot{x}}{\text{fl. } y\dot{x}}$, and therefore BAD × BC or fl. $y\dot{x} \times x - \frac{\text{fl. } y\dot{x}\dot{x}}{\text{fl. } y\dot{x}} = y \times qq$; that is, x fl. $y\dot{x}$ — fl. $y\dot{x}\dot{x} = qqy$. This in fluxions is $\dot{x}$ fl. $y\dot{x} + y\dot{x}\dot{x} - y\dot{x}\dot{x} = qq\dot{y}$; that is, $\dot{x}$ fl. $y\dot{x} = qq\dot{y}$. This put into fluxions making $\dot{x}$ conſtant, will be $y\dot{x}\dot{x} = qq\ddot{y}$; multiply by $\dot{y}$, then $y\dot{y}\dot{x}^2 = qq\dot{y}\ddot{y}$, and the fluent is $y^2\dot{x}^2 = qq\dot{y}^2$, and $y\dot{x} = q\dot{y}$, which is the property of the logarithmic curve, whoſe ſubtangent is q.

But it is not any portion of the logarithmic curve which will anſwer the queſtion, but only a part at the very ſmall end, where the curve is indefinitely near the aſſymptote, which ſuppoſe to begin at A, the weight of the reſt beyond, being ſuppoſed of no conſequence. But by Cor. 2. of the laſt Prob. when a is inconſiderable, the diſtance of the center of gravity from B, that is, BC = $\left(t - \frac{\frac{1}{2}ax\dot{x}}{ty - ax} =\right) t$, and the area ABD = ty; therefore $t \times ty$, which is the ſtreſs at B, is as qqy, which is the ſtrength at B; that is, they are in a given ratio as required. And this is all the ſolution the queſtion is capable of.

PROB. XXII.

To conſtruct the ſolid of leaſt reſiſtance, for the figure of a ſhip that will ſail the faſteſt poſſible.

Let x = abſciſſa, y = ordinate, and z = curve; then (by Ex. 17, Prob. I. Sect. II. Fluxions), the equation

equation of the curve generating (by its revolution about it axis) the ſolid of leaſt reſiſtance is $yy^3\dot{x} = aa\dot{x}^4$, that is, $yy^3\dot{x} = a \times \dot{x}^4 + 2\dot{x}^2\dot{y}^2 + \dot{y}^4$. Here x one of the variable quantities is wanting. Therefore by Rule VI. Sect. I. Fluxions, (ſee Ex. 39.) aſſume $\frac{s\dot{y}}{a} = \dot{x}$, and expunging x, we have $aasy = s^4 + 2aass + a^4$. Whence $y = \frac{s^3}{aa} + 2s + \frac{aa}{s}$ and $\dot{y} = \frac{3s\dot{s}s}{aa} + 2\dot{s} - \frac{aa\dot{s}}{ss}$, therefore $\frac{s\dot{y}}{a}$ or $\dot{x} = \frac{3s^3\dot{s}}{a^3} + \frac{2s\dot{s}}{a} - \frac{a\dot{s}}{s}$. Whence the fluent is $x = \frac{3s^4}{4a^3} + \frac{ss}{a} - a \times 2.30258$ log. s. Therefore y being known s will be known (by the equation $y = \frac{s^3}{aa} + 2s + \frac{aa}{s}$), and conſequently x by the laſt equation.

Fig. 103.

Now for conſtructing the curve, ſuppoſe $x = 30$ feet, the length of the midſhip, and $y = 10$ feet, half the greateſt breadth of the ſhip; then from the two equations,

$$\frac{3s^4}{4a^3} + \frac{ss}{a} - a \times 2.3025 \text{ log. } s = 30,$$

$$\text{and } \frac{s^3}{aa} + 2s + \frac{aa}{s} = 10,$$

we ſhall find $s = .519543$, and $a = .125311$: for taking s at pleaſure, finds $a = \frac{ss}{\sqrt{10s - a}}$. And by correcting the aſſumption, a and s are found by degrees, as above. Now the value of a being ſubſtituted in the equations above, gives theſe two following equations,

Fig. 103.

1. $x = \frac{s^4}{.002623} + \frac{ss}{.1253} - .2885 \log. s.$

2. $y = \frac{s^3}{.0157} + 2s + \frac{.0157}{s}.$

Inſtead of .2885 log. s, you may take .1442 log. ss. When x or y is a minimum, we ſhall find $s = \frac{a}{\sqrt{3}}$, whence $x = \frac{1}{1.2} a - a \times 2.3025 \log. \frac{a}{\sqrt{3}}$: And $y = \frac{16a}{3\sqrt{3}}$; or $x = .38125$, and $y = .385858$.

From the foregoing equations, the values of x, s, and y, will be found as in the following Table; by the help of which numbers, the curve may be eaſily conſtructed.

ss	x	y	ss	x	y	ss	x	y
.03630	1	.904	.15883	11	4.869	.22393	21	7.730
.05944	2	1.485	.16643	12	5.179	.22948	22	7.993
.07650	3	1.957	.17372	13	5.483	.23489	23	8.253
.09057	4	2.390	.18071	14	5.780	.24020	24	8.510
.10290	5	2.793	.18747	15	6 072	.24537	25	8.764
.11400	6	3.174	.19401	16	6.360	.25045	26	9.016
.12420	7	3.538	.20033	17	6.641	.25544	27	9.265
.13363	8	3.886	.20643	18	6.917	.26033	28	9.511
.14252	9	4.224	.21244	19	7.191	.26514	29	9.756
.15080	10	4.552	.21827	20	7.463	.26986	30	10.

The proceſs is thus; for the given value of x, find s by the firſt equation, which would be a quadratic equation, if the logarithmic quantity was wanting. Therefore it will be the eaſieſt way to find s by trial and error. When s is had, y will be found directly.

PROB.

Fig.

PROB. XXIII.

If AA *and* BB *be the poſition of the ſails of a wind-* 104.
mill, making an angle of $54\frac{1}{2}$ *with the axis of the mill* XC; *the arms of the ſails being perp. to the horizon at* C. *To determine the angle* WCX *of the winds direction* WC, *ſo as to have the greateſt force on the ſails.*

Here we ſuppoſe BB to be the top of one ſail, and AA the bottom of the other; and that BCA = an angle of twice $54\frac{1}{2}$. The common ſituation of a windmill is that XC is the direction of the wind; here WC is the direction.

Suppoſe r = CA = radius; and put s = AE or BE the S. ACE $54\frac{1}{2}$. S. WCE = z = OX, CE = b, CO = x. Then (Trig. I. 5.) $\frac{sz+bx}{r}$ = An the S. ACW, and (ib. 6.) $\frac{sz-bx}{r}$ = BF. Then (Mechanics, CII.) An^2 = force on the ſail CA, and (ib. Cor. 2.) A$n^2 \times b$ = force of the wind on the ſail SA, to move it in direction CH. Likewiſe BF$^2 \times b$ = force on the ſail CB to move it in the ſame direction CH. But the ſum of theſe is a maximum. That is, $\frac{sszz+2szbx+bbxx}{rr}b + \frac{sszz-2szbx+bbxx}{rr}b$ = max. that is, $\frac{2sszz+2bbxx}{rr}b$ = max. and $sszz + bbxx$ = max. or $ssrr - ssxx + bbxx$ = max. (becauſe $rr - xx = zz$), therefore $-2ssx\dot{x} + 2bbx\dot{x} = 0$, and $bbx = ssx$, whence $x = 0$, and therefore the wind has the greateſt force poſſible upon the ſails of a windmill, to turn them round when it blows in the ſame direction with the axis of the mill XC.

SCHOL.

Fig. 104.

SCHOL.

If AE = CE, (that is, if the ſail makes an angle of 45 with the axis), then the ſum of the forces $sszz + bbxx$, will be $sszz + ssxx$, becauſe $b = s$. Therefore $sszz + ssxx$ = max. and $zz + xx$ = max. that is, $rr - xx + xx$ = max. = rr, whoſe fluxion is 0, and rr the ſum of the forces is a ſtanding quantity. And therefore the force will be the ſame in all directions, between AC and BC.

PROB. XXIV.

Having given the angle of the ſails of a windmill with the axis $54°\frac{1}{2}$, *the velocity of the wind* 14 *feet in a ſecond, and the time of one revolution of the ſails* 6 *ſeconds. To find that point of the arm or ſail, which is not at all acted on by the wind.*

105. Let AB be the top of the ſail, draw AD for the way of the wind, perp. to BD the way of the ſail, and let AB = b, the breadth of the ſail. S. DAB, $54\frac{1}{2} = s$, its coſine = p, velocity of the wind = $v = 14^f$, and time of the ſails revolution = $t = 6''$, x = CA, A being the point required, $c = 3.1416$.

Now that the wind may not act on the ſail, the point B muſt move through BD, in the ſame time the wind moves through AD. For then the ſame particle of air will ſlide along AB without acting on it. Now if AB = b, BD = sb, and AD = pb; then $2cx$ = circumference deſcribed by A or B in one revolution. And $v : 1'' :: pb$ (AD) : $\frac{pb}{v}$ = time of deſcribing AD by the wind. Alſo $2cx : t$ $(6'') :: sb$ (BD) : $\frac{tsb}{2cx}$ = time of deſcribing BD by the

Fig. 105.

the motion. Whence $\frac{pb}{v} = \frac{tsb}{2cx}$, and $2cpbx = tsbv$, whence $x = \frac{tsv}{2cp} = 18.74$.

Cor. *Hence the ſails of a windmill ſhould never be made longer than* x; *for if they be, what is beyond will ſtrike the back wind, and retard their motion. Nor ſhould they be made quite ſo long; for a part near the end will be in a manner uſeleſs.*

PROB. XXV.

To find the weight of a pillar of marble infinitely high, whoſe baſe is B.

Let r = radius of the earth, x = any height above it, and x' any ſmall height, and Bx' = weight of a ſmall part of the cylinder or pillar, at the earth's ſurface. And by the laws of gravity $\frac{1}{rr} : Bx' :: \frac{1}{\overline{r+x}^2} : \frac{rrBx'}{\overline{r+x}^2}$, the weight of the part Bx', at the height x. Therefore the fluxion of the weight is $\frac{rrB\dot{x}}{\overline{r+x}^2} = rrB\dot{x} \times \overline{r+x}^{-2}$; and the fluent is $-rrB \times \overline{r+x}^{-1}$ or $\frac{-rrB}{r+x}$; and corrected, the true fluent is $\frac{rrB}{r} - \frac{rrB}{r+x}$, for the weight of the pillar whoſe height is x. That is, the weight $= rrB \times \frac{1}{r} - \frac{1}{r+x} = rrB \times \frac{x}{r \times r+x} = rB \times \frac{x}{r+x} \times$ ſpec. gravity of marble.

Therefore

Fig. 105. Therefore when x is infinite, the weight of the infinite pillar of marble is $= rB \times \frac{x}{x} = rB \times$ ſpec. gravity.

Cor. *The weight of an infinite pillar of any matter is = weight of the ſolid hyperboloid, generated by an hyperbola revolving about the aſymptote; whoſe baſe is* B, *and inſcribed parallelogram is rd, where* $d =$ *radius of the baſe of the pillar, and the gravity undiminiſhed.*

By Ex. 9. Prob. XIV. Sect. II. Fluxions.

PROB. XXVI.

To find the ſtrength of a piece of iron and a piece of wood, of the ſame length, and of equal weight. Alſo to find the weights, when the ſtrength is the ſame.

106. 1. We ſhall ſuppoſe theſe to be two ſquare pieces of equal length. Let A be a beam of iron, B a beam of wood of the ſame bigneſs, C a beam of iron of the ſame weight as B. Let $y =$ ſide of the ſquare of A and B, $x =$ that of C, $w =$ the ſtrength of wood, $n =$ ſtrength of iron, s and $t =$ ſpecific gravities of wood and iron. Now that B and C may have the ſame weight, $syy = txx$; and ſtrength of the wood B : ſtrength iron A :: $w : n$; and ſtrength of iron A : ſtrength of iron C :: $y^3 : x^3$. Whence ſtrength of wood B : ſtrength of iron C :: $wy^3 : nx^3$ :: (becauſe $y = \sqrt{\frac{txx}{s}}$) $\frac{wtxx}{s}\sqrt{\frac{txx}{s}} : nx^3 :: \frac{wt}{s}\sqrt{\frac{t}{s}} : n :: wt\sqrt{t} : ns\sqrt{s}$. That is, ſtrength of the wood B : to the ſtrength of the iron C, of the ſame weight :: $\frac{w}{s\sqrt{s}} : \frac{n}{t\sqrt{t}}$.

Suppoſe

Suppoſe elm or aſh whoſe ſpecific gravity is .8, and iron is 7.6. Alſo $8\frac{1}{2}$ and 107 are the ſtrength of this wood and iron. Therefore $w = 8\frac{1}{2}$, $n = 107$, and $s = .8$, $t = 7.6$. Therefore ſtrength of the wood : ſtrength of iron of the ſame weight; as $\frac{8\frac{1}{2}}{.8\sqrt{.8}}$ to $\frac{107}{7.6\sqrt{7.6}}$ or as 11.8 to 5.1; therefore this wood is twice as ſtrong for the weight as iron. Fig. 106.

2. Suppoſe B and C of the ſame ſtrength, in which caſe $wy^3 = nx^3$, and $y = x\sqrt[3]{\frac{n}{w}}$. But weight of B : weight of C : : syy : txx : : $sxx\sqrt[3]{\frac{nn}{ww}}$: txx : : $s\sqrt[3]{\frac{nn}{ww}}$: t : : $\frac{s}{\sqrt[3]{ww}}$: $\frac{t}{\sqrt[3]{nn}}$.

Suppoſe the wood to be oak, then $s = .8$, $t = 7.6$, and $w = 11$, $n = 107$. Therefore the weight of oak : to the weight of iron of the ſame ſtrength : : is as $\frac{.8}{\sqrt[3]{11^2}}$: to $\frac{7.6}{\sqrt[3]{107^2}}$: : that is, as 16 to 34; therefore a piece of iron of the ſame ſtrength as a piece of oak, weighs as much more.

PROB. XXVII.

Suppoſe a ſtream of water to run down the plane AC, *whoſe elevation above the horizon* GCH *is* 30°; 107. *and the perp.* BC *be* 8 *feet.*

To find the radius of the water wheel DBF, *ſo that the effect of the wheel, driven by this ſtream of water, ſhall be a maximum.*

Let E be the center of the wheel, and draw EB, ED, perpendicular to AC, CG, and draw CE. Since the angles at D and B are right, the angle DEB = ACH = CAP, ſuppoſing APF a horizontal line; and the angle CEB = $\frac{1}{2}$CAP = 15°.

Let

Fig. 107. Let EB $= x$, t = tan. BEC = tan. 15°, AC $= a$, PC $= p$, S. PAC $= S$; then radius (1) : EB (x) : : tan. BEC (t) : tx = CB; and AB $= a - tx$. Alſo AC (a) : CB (p) : : AB ($a - tx$) : $p - \frac{ptx}{a}$ = the depth at B. But the effect of the wheel, will be as the velocity of the water on the floats at B, and as the radius of the wheel EB, or as $x \times \sqrt{p - \frac{ptx}{a}}$; therefore $x\sqrt{p - \frac{ptx}{a}}$ = max. and $xx \times \overline{a - tx}$ or $axx - tx^3$ = max. In fluxions $2ax\dot{x} - 3tx^2\dot{x} = 0$, or $2a = 3tx$, whence $x = \frac{2a}{3t} = \frac{2p}{3ts}$, becauſe $a = \frac{p}{s}$, therefore $x = 39.8$ feet.

Cor. *If* PAC *be a right angle, or the water deſcend perpendicularly down* PC; *then* t = *tan.* 45° = 1, a = PC $= p$; *then* $x = \frac{2}{3}a = \frac{2}{3}$PC $= 5\frac{1}{3}$ *feet.*

PROB. XXVIII.

Given the length of a cylinder CB, *and its inclination to the horizon; to find the time of its falling to the ground* CE.

108. The time of its falling will be the ſame as the time of a body's falling through the arch DE of a circle, whoſe radius CD $= \frac{2}{3}$ CB, D being the center of oſcillation. Therefore to find the time of deſcent of D through the arch DE.

Draw DF, gI, perp. and Db parallel to AC, where AC is perp. to the horizon; and let CD $= r$, AF $= a$, FC $= s$, D$b = x$, gI $= y$, DI $= z$, $b = r + s$, t = time of deſcending through DI, $d = 16.1$ feet; then $2d$ = velocity acquired by a falling body in 1″. Alſo $\sqrt{d}$ (height) : $2d$ (velocity) : : $\sqrt{x}$

$\sqrt{x}$ (height) : $\sqrt{2dx}$ = velocity at b or I. Alſo, Fig. 108.
2d (ſpace) : 1″ (time) : : x' (ſpace) : $\frac{x'}{2d}$ = time of deſcribing x'. But (Fluxions, Prob. I. Sect. III.) $\dot{t} \propto \frac{\dot{s}}{v}$, and therefore $\frac{x'}{2d}$ (time) : $\frac{x'}{2d}\left(\frac{\text{ſpace}}{\text{vel.}}\right)$: : t' (time) : $\frac{\dot{z}'}{2\sqrt{dx}}$; therefore $\dot{t} = \frac{\dot{z}}{2\sqrt{dx}}$. But $\dot{z} : \dot{x} : : r : y$, and $\dot{z} = \frac{r\dot{x}}{y}$, and $yy = \overline{a+x} \times \overline{b-x}$. Therefore $\dot{t} = \frac{r\dot{x}}{2y\sqrt{dx}} = \frac{r\dot{x}}{2\sqrt{dx}\sqrt{\overline{a+x} \times \overline{b-x}}} = \frac{r}{2\sqrt{d}} \times \frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{\overline{a+x} \times \sqrt{b-x}}} = \frac{r}{2\sqrt{d}} \times \frac{x^{-\frac{1}{2}}\dot{x}}{\sqrt{ab+2sx-xx}}$. And the fluent found by infinite ſeries, will give t.

And if t be given, then by reverſion of ſeries, x may be found.

PROB. XXIX.

Suppoſing a ſhip without lee way; and WS *be the direction of the wind,* SA *the poſition of the ſail;* SD *the poſition of the keel, or ſhip's way. Any one of the angles* WSA, ASD, *or* WSD, *being given; to find the reſt, ſo that the ſhip may gain the moſt poſſible to windward in a given time.* 109.

Let the S.WSA = s, its coſ. ASG = c, S.ASD = x, coſ. WSD or S.DSG = z, ſuppoſing SGP perp. to WS; r = radius.

Then (by Cor. 5. Prop. CVI. Mechanics,) the velocity of the ſhip to windward is as S.WSA × $\sqrt{\text{S.ASD}}$ × coſ. WSD; that is, as $sx^{\frac{1}{2}}z$. Then,

Fig. 109. 1. Suppose WSA or s to be given; then $x^{\frac{1}{2}}z$, or $xzz = $ a maximum. Whence $\dot{x}zz + 2z\dot{z}x = 0$, and $z\dot{x} + 2x\dot{z} = 0$. But (by Prop. VI. B. I. Trigonometry,)

$\frac{c\sqrt{rr-xx}-sx}{r} = z$, and $\frac{\frac{-cx\dot{x}}{r\sqrt{rr-xx}} - s\dot{x}}{r} = \dot{z}$,

therefore $z\dot{x} + 2x \times \frac{\frac{-cx\dot{x}}{r\sqrt{rr-xx}} - s\dot{x}}{r} = 0$,

and $z = \frac{\frac{2cxx}{r\sqrt{rr-xx}} + 2sx}{r} = \frac{c\sqrt{rr-xx}-sx}{r}$,

and $\frac{2cxx}{\sqrt{rr-xx}} + 3sx = c\sqrt{rr-xx}$, and $2cxx + 3sx\sqrt{rr-xx} = crr - cxx$, or $3sx\sqrt{rr-xx} = crr - 3cxx$. involved is $9ssrrxx - 9ssx^4 = c^2r^4 - 6ccrrxx + 9ccx^4$, which reduced (putting $rr = ss + cc$), then $9x^4 - 9ssxx - 6ccxx + ccrr = 0$; which equation contains 4 roots, two affirmative and two negative. And the angle DSG will be about double to ASD, or their sines.

Hence if WSA $= 0, 30, 45, 60, 90$; then ASD will be $= 35, 22, 16, 10, 0$, respectively.

2. Suppose x or the angle ASD to be given; then $sz = $ max. and $s\dot{z} + z\dot{s} = 0$, or $z\dot{s} = -s\dot{z}$. Let $d = $ cos. ASD $= \sqrt{rr-xx}$, and $c = \sqrt{rr-ss} = $ cos. ASW $=$ S. ASG. Therefore instead of $\frac{c\sqrt{rr-xx}-sx}{r} = z$, we have $\frac{d\sqrt{rr-ss}-sx}{r} = z$,

and $\dot{z} = \frac{-ds\dot{s}}{r\sqrt{rr-ss}} - \frac{x\dot{s}}{r} = 0$, whence $z\dot{s} = \frac{ds\dot{s}s}{r\sqrt{rr-ss}} + \frac{xs\dot{s}}{r}$, and $z = \frac{dss}{r\sqrt{rr-ss}} + \frac{xs}{r} = \frac{d\sqrt{rr-ss}-sx}{r}$, therefore $\frac{dss}{\sqrt{rr-ss}} + 2sx = d\sqrt{rr-ss}$, and $dss + 2sx\sqrt{rr-ss} = drr - dss$, and

2sx

Fig. 109.

$2sx\sqrt{rr-ss} = drr - 2dss$, and $4ssrrxx - 4xxs^4 = dd \times rr - 2ss^2$, and puting $rr - xx$ for dd, and reducing, we have $4rrs^4 - 4r^4ss - r^4x^2 + r^6 = 0$, and $4s^4 - 4rrss + r^4 = rrxx$, and extracting the root, $2ss - rr = rx$, or $rr - 2ss = rx$. But (by Trigonometry) $\frac{rr-2ss}{r}$ = is the cosine of 2WSA; and since x = S.DSA = cos. WSA + DSG, therefore 2WSA = WSA + DSG, therefore WSA = DSG; and either of them = $\frac{1}{2}$ comp. of ASD.

3. If WSD or z be given, then put n = S.WSD. Then $s\sqrt{x}$ or ssx = max. and $2ss\dot{x} + ss\dot{x} = 0$, and $2\dot{x}s = -s\dot{x}$. But (by Trigonometry) $\frac{n\sqrt{rr-ss}-zs}{r} = x$, and $\frac{-nss}{r\sqrt{rr-ss}} - \frac{z\dot{s}}{r} = \dot{x}$, whence $2x\dot{s} = \frac{nsss}{r\sqrt{rr-ss}} + \frac{zs\dot{s}}{r}$, and $x = \frac{nss}{2r\sqrt{rr-ss}} + \frac{zs}{2r} = \frac{n\sqrt{rr-ss}-zs}{r}$, and $\frac{nss}{2\sqrt{rr-ss}} + 3zs = 2n\sqrt{rr-ss}$ and $nss + 3zs\sqrt{rr-ss} = 2nrr - 2nss$, and $3zs\sqrt{rr-ss} = 2nrr - 3nss$, and $zz \times 9rrss - 9s^4 = 4n^2r^4 - 12n^2r^2ss + 9n^2s^4$, and putting $rr - nn$ for zz, and reducing $9s^4 - 9rrss + 4rrnn = 0$, $-3nn$ from which equation s will be found.

If WSD be between 45 and 60 degrees, then WSA is nearly twice ASD.

Fig.

PROB. XXX.

110. *There is a river* FKD *runs due eaſt, and ſeveral ſwimmers of equal ſwiftneſs, ſet off from a point* A *in the north ſide; and move in all directions, or all degrees of the arch* KBD. *To find which ſwimmer will land on the oppoſite ſide, the furtheſt up the river.*

Let $b =$ breadth AD or AB, and let b repreſent the ſwimmers motion, and $d =$ BC the currents motion, AE $= x$, EB $= y$. Then the ſwimmer going off in direction AB, by the compound motion, goes in the line AC, and lands at H, ſo that HD is a minimum, by ſimilar triangles, AE (x) : EC ($d - y$) : : AD (b) : DH $= \frac{b}{x} \times \overline{d-y} =$ min. and $\frac{d-y}{x} =$ min. whence $\frac{-x\dot{y} - \overline{d-y} \times \dot{x}}{xx} = 0$, and $-x\dot{y} = \overline{d-y} \times \dot{x}$, but $xx + yy = bb$, and $x\dot{x} = -y\dot{y}$, or $-\dot{y} = \frac{x\dot{x}}{y}$; therefore $\frac{xx\dot{x}}{y} = d\dot{x} - y\dot{x}$, and $xx = dy - yy = bb - yy$, therefore $dy = bb$, and $y = \frac{bb}{d}$.

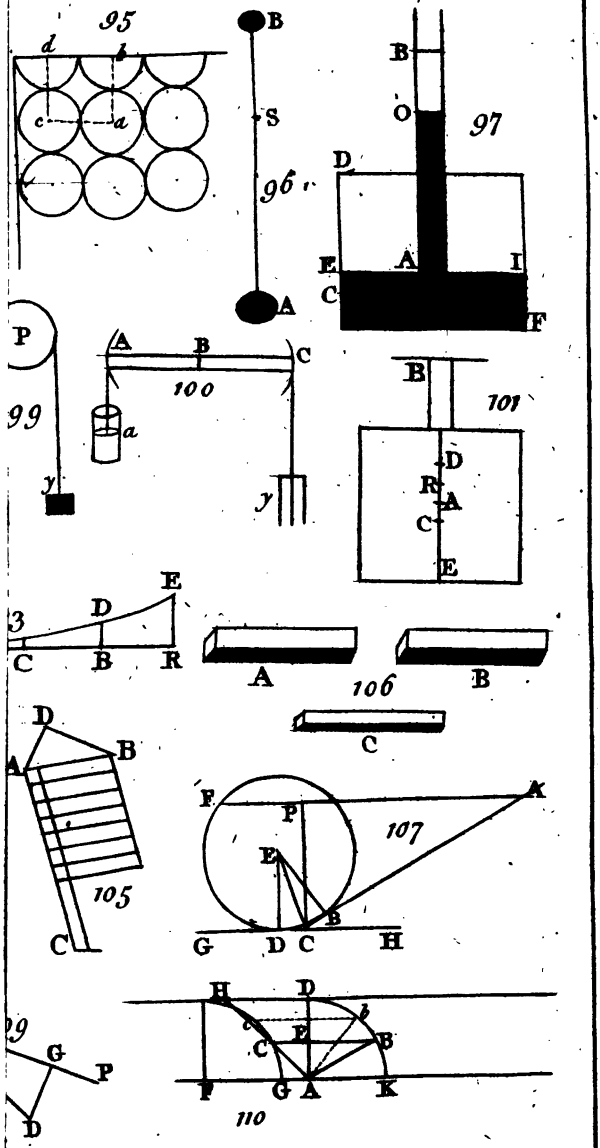
Cor. 1. *The direction* AC *will touch the circle* GCH, *equal to the circle* KBD; *for all the lines* GK, CB, *cb, &c. are equal; d being greater than b.*

Cor. 2. *If b is equal or greater than d, the ſwimmers direction is extremely near* AK.

Cor. 3. HD $= \sqrt{dd - bb}$.

Cor. 4. *If the time of paſſing through* AB *be* $= t$; *the time of arriving at* H *will be* $\frac{dt}{\sqrt{dd-bb}}$.

For



Pl. VIII. *pa.468*

UNIV. OF MICH.

For x $\left(\sqrt{bb - \frac{b^4}{dd}}\right) : t :: b :$ time ſought. Fig.

SCHOL.

The river will be croſſed in the leaſt time, by moving in the perpendicular direction AD.

PROB. XXXI.

Let CD *be an exponential curve, whoſe equation is* $y = x^x$, *putting* AB $= x$, BD $= y$. *To find the area, and the leaſt ordinate.* III.

1. Let X = hyp. log. x, Y = hyp. log. y. Then ſince $y = x^x$, therefore $Y = xX$; and by the nature of logarithms (ſee Schol. 2. Prob. II. Sect. II. Fluxions,) $y = 1 + \dot{Y} + \frac{Y^2}{2} + \frac{Y^3}{6} + \frac{Y^4}{24}$ &c. $= 1 + xX + \frac{x^2X^2}{2} + \frac{x^3X^3}{3}$, &c. and the fluxion of the area $y\dot{x}$ or $x^x\dot{x} = \dot{x} + X\dot{x}\dot{x} + \frac{X^2\dot{x}^2\dot{x}}{2} + \frac{X^3\dot{x}^3\dot{x}}{6} + \frac{X^4\dot{x}^4\dot{x}}{24}$ &c. And the fluent $= x + \frac{\dot{x}\dot{x}X}{2} + \frac{\dot{x}^3X^2}{6} + \frac{\dot{x}^4X^3}{24}$ &c. $- s$. And proceeding by Rule 8, Sect. I. Fluxions, $s = \frac{x^2}{4} + \frac{x^3X}{9} + \frac{x^4X^2}{32}$ &c. $- t$. And $t = \frac{x^3}{27} + \frac{x^4X}{43}$ &c. $- u$. And $u = \frac{x^4}{4^4}$, and ſo on; therefore fluent $x^x\dot{x} = x + \frac{x^2X}{2} + \frac{x^3X^2}{6} + \frac{x^4X^3}{24}$ and $- s + t - u$ &c. Or the area

Fig. III.

$$= x + \frac{xxX}{2} + \frac{x^3X^2}{6} + \frac{x^4X^3}{24} \text{ &c.}$$
$$- \frac{x^2}{4} - \frac{2x^3X}{18} - \frac{x^4X^2}{32}$$
$$+ \frac{x^3}{27} + \frac{x^4X}{4^3}$$
$$- \frac{x^4}{4^4}$$

$$= x + \frac{xY}{2} + \frac{xY^2}{6} + \frac{xY^3}{24} \text{ &c.}$$
$$- \frac{x^2}{4} - \frac{2x^2Y}{18} - \frac{x^2Y^2}{32}$$
$$+ \frac{x^3}{27} + \frac{x^3Y}{4^3}$$
$$- \frac{x^4}{4^4}$$

But when $x = 0$, the area $= 0$, and $X = -$ infinity, but xX, x^2X^2, x^3X^3, &c. are each $= 0$. Therefore the area above is correct.

Cor. *When* $x = 1$, *then* $X = 0$, *and the area* $= x - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5}$ &c.

2. For the leaſt ordinate, when y or x^x is a minimum, then y or xX = minimum; whence $x\dot{X} + X\dot{x} = 0$, that is, $\frac{x\dot{x}}{x} + X\dot{x} = 0$, and $X = -1$, the hyp. log. x; and the common log. $x = \frac{-1}{2.3025} = .434294 = -1.565706$, and $x = .36788$, and $y = .6922$.

PROB. XXXII.

If two bodies p, q, be ſuſpended by ropes on the axis in peritrochio; to find the preſſure on the axis, when the bodies are left at liberty to deſcend.

Let

Let rad. wheel $Ab = a$, rad. of the axis $CB = b$, $b = 16.1$, the height deſcended in a ſecond by gravity; and ſuppoſe p outweighs q on the machine, divide the weight p into two parts n and r, ſo that CA or $Cb \times r = CB \times q$, or $r = \frac{bq}{a}$, then $p = n + \frac{bq}{a}$; and hence it is plain the weight r acting at A, and the weight r acting at b (or q acting at B) will keep the machine in equilibrio, and it will remain at reſt. Suppoſe then (inſtead of q) that the weight $n + r$ hangs at A, and r at b; the force that gives motion to the wheel is the part n acting at A. Now to find the ſpace x through which theſe bodies are drawn in a ſecond, by that force. When the time is given, the ſpace is as the force divided by the body (by Mechanics); therefore $\frac{n}{n}\left(\frac{\text{force}}{\text{body}}\right) : b$ (ſpace) $:: \frac{n}{p+r}\left(\frac{\text{force}}{\text{body}}\right) : x = \frac{nb}{p+r} = \frac{p-r}{p+r}b$, the ſpace through which r is raiſed upwards. But when the time is given, the ſpace is as the force that generates it, b (ſpace) : r (force) : : x (ſpace) : $\frac{rx}{b}$ = force acting at A and b, that drives r up. But the force acting at B to drive q up, is $\frac{a}{b} \times \frac{rx}{b}$; and both theſe forces muſt be ſupported by the axis at C, or elſe they could not act againſt one another; therefore $\frac{rx}{b} + \frac{arx}{bb}$ is the preſſure upon the axis by the motion of the machine; to which we muſt add the dead weight which the machine ſupported when it was

Fig. 112.

Fig. 112. at reſt; and this additional weight is $q + r$, becauſe r (ſuſpended at A) balances q (ſuſpended at B). Whence the whole preſſure upon the axis is $\frac{a+b}{b} \times \frac{rx}{b} + q + r = \frac{a+b}{bb} \times \frac{p-r}{p+r} rb + q + r = \frac{a+b}{b} r \times \frac{p-r}{p+r} + q + r$, where $r = \frac{bq}{a}$.

EXAM.

Suppoſe $a = 5$, $b = 2$, $p = 4$, $q = 9$, then $r = 3.6$; and the weight $= \frac{25.2}{2} \times \frac{4-3.6}{4+3.6} + 9 + 3.6 = .66 + 9 + 3.6 = 13.26$.

PROB. XXXIII.

113. *In the log. ſpiral* AfB, *if the ordinates* CA, CB, *be given, and the angle* ACB. *To find the angle of the ſpiral, or angle* B.

Draw the arch of the circle Apq; and let there be an indefinite number $(n - 1)$ of mean proportionals Cc, Cd, Ce, &c. between AC and BC. And let AC $= a$, BC $= b$, C$c = x$, arch Apq $= H$; then $a : x : \frac{xx}{a} : \frac{x^3}{aa} : \frac{x^4}{a^3} : \frac{x^5}{a^4} \ldots \frac{x^n}{a^{n-1}} = b$; whence C$c$ is the firſt of $n - 1$ mean proportionals, between AC and CB: and the arch Ar $= \frac{1}{n}$ of the arch Apq. Then $\frac{x^n}{a^n} = \frac{b}{a}$, and log. $\left|\frac{x}{a}\right|^n$ $=$ log. $\frac{b}{a}$ $=$ log. $b -$ log. $a =$ B $-$ A, (putting B, A, for theſe logs.) that is $n \times$ log. $\frac{x}{a} =$ B $-$ A,

$-A$, and log. $\frac{x}{a} = \frac{B-A}{n} = L$; whence $\frac{x}{a}$ Fig. 113.
$=$ num. of the log. $L = 1 + \frac{L}{m} + \frac{L^2}{2m^2} + \frac{L^3}{6m^3}$ &c. by the nature of logarithms: where $m = 1$, for the hyp. logarithms, or $m = .43429$ for the common logarithms. Whence $x = a + \frac{aL}{m} + \frac{aL^2}{2m^2}$ &c. but as n is exceeding great, $x = a + \frac{aL}{m}$, and $x - a = \frac{aL}{m}$, and $Ar = \frac{H}{n}$. Let $n =$ 1000 000, then will $rc = \frac{a}{m} \times \frac{B-A}{1000\,000}$, and $Ar = \frac{H}{1000\,000}$; and the triangle Arc, may be looked on as a right angled plain triangle. And it will be, as $cr : Ar ::$ rad. (1) : $\frac{Ar}{cr} =$ tan. $\angle$ Acr or $\angle$ B required $= \frac{mH}{a \times B - A}$.

Cor. *When the angle at* B *is found, it will be, as cof.* B : BC : : *rad.* (1) : *length of the ſpiral* BAQC, *&c.*

For $\frac{BC \times \text{rad.}}{\text{cof. B}}$ = tangent at B, and (Prop. IV. Log. Spiral,) that tangent = whole length of the ſpiral.

PROB. XXXIV.

To find a point P *within a parallelogram; through* 114. *which, if two lines* AB, CD, *be drawn parallel to the ſides, the ſegments ſhall be in continual proportion,* AP : CP : PB : PD.

Let

Fig. 114. Let AP $= x$, PC $= y$, then PB $= \frac{yy}{x}$, PD $= \frac{y^3}{xx}$, AB $= a$, CD $= b$. Then $x + \frac{yy}{x} = a$, or $xx + yy = ax$. And $y + \frac{y^3}{xx} = b$, or $y \times \overline{xx+yy} = bxx$. Hence $y \times ax = bxx$, and $ay = bx$, and $y = \frac{bx}{a}$; therefore $ax = xx + yy = xx + \frac{bbxx}{aa}$, and $a^3x = aaxx + bbxx$, or $aax + bbx = a^3$, and $x = \frac{a^3}{aa+bb}$, and $y = \frac{aab}{aa+bb}$.

PROB. XXXV.

A ſhip ſailed directly ſouth, with a uniform motion, from lat. 37°, *to lat.* 37° 48′, *in* 22 *hours; while the vane all the while was kept in the ſame poſition,* N. E. *by* E. *by a wind which moved* 20 *miles, or minutes an hour. I demand the true point of the wind.*

115. Let $m = 48'$, $s = \frac{48}{22} = \frac{m}{22}$, the ſhip's motion in an hour, $w = 20$, the wind's motion in an hour. Let AB be the ſhip's way, AC the poſition of the vane, AD the direction of the wind. Draw CD parallel to AB, and DB parallel to AC. Becauſe the vane is at reſt as it moves through the wind, it acts upon neither ſide; and therefore the particles of wind that were in contact with the vane at A, in the poſition CA; will alſo be in contact with the vane at D, in the poſition DB; becauſe theſe particles neither approach nor recede from the vane; otherwiſe they would move the vane one way or other.

The

The angle CAB is given, and therefore DBA is given, and in the triangle ADB, AD (w) : AB (s) :: S.ABD (5 points or 56° 15′) : S.ADB or CAD = 5° : 12′. From CAB, 123° 45′, take 5° 12′ = CAD, then DAB = 118° 33′, the point of the wind. Fig. 115.

Cor. *Hence this rule, as the velocity of the wind* AD : *to the velocity of the ſhip* AB : : *ſo the ſine of the angle between the ſhip's way and the poſition of the vane : to the ſine of the difference between the true and apparent place of the wind.*

PROB. XXXVI.

The circumference of the circle ABQ *is* 21; *a ſtring is put about it, and knotted at* P, *whoſe length is* 25. *To find the area deſcribed by drawing the knot* P *round about.* 116.

It is plain that P will deſcribe a circle, whoſe center is C, the center of the firſt. Draw CD perp. to ACP, and CB perp. to BP the tangent at B. Then AD is a quadrant = $5\frac{1}{4}$. Radius CD or CB = $r = \frac{10.5}{3.1416}$. Let arch DB = a, then DBP = ſum of the arch and cotangent = $7\frac{1}{4} = n$; but the cotan. BP = $\frac{rr}{a} - \frac{a}{3} - \frac{a^3}{45rr} - \frac{2a^5}{945r^4} - \frac{a^7}{4725r^6}$ &c. Therefore $\frac{rr}{a} + \frac{2}{3}a - \frac{a^3}{45r^2} - \frac{2a^5}{945r^4} - \frac{a^7}{4725r^6} = n$. And by reverſion $a =$ 1.8493; then 21 : 360 : : 1.849 : 31.7 degrees = 31° 42′ = < DCB, or BPC. Then S.BPC (31° 42′) : CB (3.3422) : : rad. : CP = 6.551, the radius of the circle deſcribed by P. Then $CP^2 \times 3.1416 = 134.82$ the area of that circle.

PROB.

Fig.

PROB. XXXVII.

117. *If* ADF *be the arch of a circle, and ſuppoſing the arch* AD *continually evolves, beginning at* A, *the point* A *deſcribing the curve* ABG. *To find the length* AG, *and area* FAG.

Let the radius CA $= r$, arch AD $= z$. Take Dd infinitely ſmall, and draw the tangents DB, db; and make AB $= v$, Bb $= \dot{v}$, Dd $= \dot{z}$, then DB $=$ arch DA, and the angle BDb $=$ DCd, and the ſectors DCd and BDb are ſimilar; whence CD (r) : DB (z) : : Dd $(\dot{z})$: Bb $(\dot{v}) = \frac{z\dot{z}}{r}$, and v or AB $= \frac{zz}{2r}$, and AG $= \frac{\overline{\text{ADF}}^2}{2r}$.

Again, DB being perp. to AB, the fluxion of the area ABD is $= \frac{z\dot{v}}{2} = \frac{z^2\dot{z}}{2}$; and the area ABD $= \frac{z^3}{6r}$, and area AGF $= \frac{\text{AF}^3}{6r}$.

PROB. XXXVIII.

Suppoſe the value of a moor to be 4*l.* 8*s. an aker; and walling will coſt* 6*s. a rod, or* 7 *yards in length. To find the quantity of ground (in form of a ſquare,) to be taken in, ſo that the walling may be juſt equal to its value.*

Let $x =$ ſide of the ſquare in yards, $r =$ 1 rod, or 7 yards long, $p =$ price of walling a rod, $a =$ an acre, $m =$ value of it. Then $r : p$: : circumference $4x : \frac{4px}{r} =$ whole price of walling. And

$a : m :: xx : \frac{mxx}{a}$ = whole value of the ground; Fig. 117.

then, per queſt. $\frac{mxx}{a} = \frac{4px}{r}$, and $x = \frac{4pa}{rm} = \frac{24}{20 \times 7} \times \frac{4840}{4.4} = 188.6$ yards. And $\frac{\overline{188.6}^2}{4840} = 7.35$ acres.

PROB. XXXIX.

To find a number (x), by which multiplying the difference of the bung and head diameters of a ſpheroidal caſk, and then added to the leſſer will be the mean diameter, or the diameter of a cylinder equal to the caſk.

Let D = bung diameter, d = head diameter, then $\overline{D - d} \times x + d$ = mean diameter. Then the ſolidity of the caſk $= \frac{2D^2 + d^2}{3} \times .7854b = \overline{\overline{D - d} \times x + d}^2 \times .7845x$, and $\overline{\overline{D - d} \times x + d}^3 = \frac{2D^2 + d^2}{3}$, and $\overline{D - d}.x + d = \sqrt{\frac{2}{3}DD + \frac{1}{3}dd}$, and $\overline{D - d}.x = \sqrt{\frac{2}{3}DD + \frac{1}{3}dd} - d$, whence $x = \frac{\sqrt{\frac{2}{3}DD + \frac{1}{3}dd} - d}{D - d}$.

Cor. $\sqrt{\frac{2}{3}DD + \frac{1}{3}dd}$ *is the mean diameter ſought. And x is always between $\frac{2}{3}$ and $\sqrt{\frac{2}{3}}$, or between .666 and .816.*

SCHOL.

After the ſame manner $\sqrt{\frac{DD + Dd + dd}{3}}$ = mean diameter for the fruſtum of a cone. And x always between $\frac{1}{2}$ and $\sqrt{\frac{1}{3}}$, or between .5 and .578.

PROB.

Fig.

PROB. XL.

118. *To find the ſurface of the ſolid generated by a cycloid* AMG *revolving about its ſhorter axis* AR.

Let AR $= a$, AP $= x$, arch AB $= s$, PB $= v = \sqrt{ax - xx}$, PM $= y$, S $=$ ſurface required, AM $= z$. By the nature of the cycloid, $zz = 4ax$, and $2z\dot{z} = 4a\dot{x}$, or $z\dot{z} = 2a\dot{x}$, whence $\dot{z} = \frac{2a\dot{x}}{z} = \frac{2a\dot{x}}{\sqrt{4ax}} = \dot{x}\sqrt{\frac{a}{x}}$; and $y = v + s$. Then $\dot{S} = cy\dot{z} = \overline{v+s} \times c\dot{x}\sqrt{\frac{a}{x}} = c\ddot{x}\sqrt{\frac{a}{x}} \times \sqrt{ax - xx} + sc\dot{x}\sqrt{\frac{a}{x}} = c\sqrt{a} \times \dot{x}\sqrt{a-x} + c\sqrt{a} \times sx^{-\frac{1}{2}}\dot{x}$. And the fluent $S = c\sqrt{a} \times -\frac{2}{3} \times \overline{a-x}^{\frac{3}{2}} +$ fl. $c\sqrt{a} \times sx^{-\frac{1}{2}}\dot{x} = -\frac{2}{3}c\sqrt{a} \times \overline{a-x}^{\frac{3}{2}} + c\sqrt{a} \times 2sx^{\frac{1}{2}} + c\sqrt{a} \times 2a\sqrt{a-x}$; becauſe $s = \frac{a\dot{x}}{2v}$. And the fluent corrected, or S $= c\sqrt{a} \times$ into $\frac{4a+2x}{3}\sqrt{a-x} + 2s\sqrt{x}$: and the whole ſurface $= 2ca \times \overline{ABR - \frac{2}{3}a}$.

PROB. XLI.

119. *Let* ABE *be a ſemicircle whoſe center is* C; MD, DQ, *two ſides of a ſquare. And let the ſquare revolve about the circle, ſo that the ſide* DM *may always paſs through the point* E *at the end of the diameter, whilſt the other ſide* DQ *always touches the circle. To find the locus of the angular point* D.

Upon the radius CE of the given circle, deſcribe the ſemicircle CHE; and from the center C draw

draw CB to the point of contact B, which will be perp. to the tan. DB; and D being a right angle, CB and ED are parallel. Through the center O, draw OHP parallel to ED or CB; then the angle EOH = ECB; and drawing HF perp. to EF; HF and BD will be tangents to the two circles, and parallel to one another. Therefore the figures EOHF, ECBD, are similar. And since EO = $\frac{1}{2}$ EC, EF = $\frac{1}{2}$ ED = FD, and EH = DH. Make HP = HO, and draw DP. Fig. 119.

The triangles EHO, DHP, are similar and equal; for DH = EH, and PH = OH, and ∠ DHP = EHO, (they being the complements of the equal angles DHF, EHF), therefore PD = EO = PH. Whence if the circle DHNB be described with the radius PH or PD; the arches DH, EH, will be equal, because their cords are equal. And consequently if the arch DH revolves upon the equal arch EH, it will describe an epicycloid passing through D; or the point D in the square, will describe the same epicycloid, which will be the locus of all the points D.

PROB. XLII.

To find the length of the curve of the epicycloid VD. 120.

Let AB = b, AC = n, AV = d, VB = $2r$, arch WD or GE, or BE = s. Sine Dm = v, vers. Wm = x, AD = y, Do = Ee = Ff = $\dot{s}$, arch VD = z, then $d = b + 2r$, and $n = b + r$.

With the radius Ad, describe the arch *nod*. Then whilst the point of contact E is carried through the arch Ee, the line Ao is carried through the arch Ff = Ee, and therefore AD through the angle DAd. Draw CP perp. to AD. By the nature of the circle $vv = 2rx - xx$, and $yy = \overline{d - x}^2 + vv$

=

Fig. 120. $= dd - 2nx$. And $vs = r\dot{x}$. The triangles ADm and ACP are similar, also CDP and Dno: therefore AD (y) : Am $(d - x)$: : AC (n) : AP $= \frac{d-x}{y}\, n$. And DP $= y - \frac{d-x}{y}\, n = \frac{rd-nx}{y}$. Again, CD (r) : DP $\left(\frac{rd-nx}{y}\right)$: : Do $\left(\dot{s} \text{ or } \frac{r\dot{x}}{v}\right)$: $no = \frac{rd-nx}{vy}\, \dot{x}$. Also AF (b) : Ao (y) : : Ff $\left(\dot{s} \text{ or } \frac{r\dot{x}}{v}\right)$: $od = \frac{ry}{bv}\, \dot{x}$. Then $nd = \frac{ry}{bv}\, \dot{x} + \frac{rd-nx}{vy}\, \dot{x} = \frac{ryy + brd - bnx}{bvy}\, \dot{x} = \frac{2r-x}{bvy}\, dn\dot{x}$. Also $n\mathrm{D} = \dot{y} = \frac{-n\dot{x}}{y}$; whence $\mathrm{D}d^2$ or $\ddot{x}^2 = \frac{\overline{2r-x}^2}{b^2v^2y^2}\, d^2n^2\dot{x}^2 + \frac{n^2\dot{x}^2}{yy} = \frac{\overline{2r-x}^2.dd - bbvv}{b^2v^2y^2}\, n^2\dot{x}^2 = \frac{\overline{2r-x}.dd + bbx}{bbyyx}\, n^2\dot{x}^2 = \frac{2rdd - 4nrx}{bbyyx}\, n^2\dot{x}^2 = \frac{dd-2nx}{bbyyx} \times 2rn^2\dot{x}^2 = \frac{2rn^2\dot{x}^2}{bbx}$. Therefore $\dot{z} = \frac{n\dot{x}}{b}\sqrt{\frac{2r}{x}}$, and $z = \frac{2n}{b}\sqrt{2rx} = \frac{2b+2r}{b} \times$ cord WD.

PROB. XLIII.

121. *To find the place of a planet in its orbit, or the angle at the other focus* F, *having the distance* FN *given.*

Let S be the sun, AC $= a$, CE $= c$, CF $= n$, SN $= y$, FN $= v$, M $=$ mean motion, F $=$ motion round F, N the planet's place, $\dot{x} =$ perp. from n upon SN, and FN.

Then

Then (by Prop. II. Sect. V. Aſtronomy,) $\dot{M} = \frac{y\dot{x}}{c}$, and $\dot{F} = \frac{a\dot{x}}{v}$. And M = area $\frac{ASN}{c}$. To find the fluent of $\frac{a\dot{x}}{v}$; draw FD perp. to the tangent NT; then (by Cor. 2. Prop. XXXVI. Ellipſis,) $\sqrt{vy} : c :: v : FD = \frac{cv}{\sqrt{vy}}$; and the triangles Non, NDF, are ſimilar, and ND $\left(\sqrt{vv - \frac{ccv}{y}}\right)$: DF $\left(\frac{cv}{\sqrt{vy}}\right)$ :: $\dot{v} : \dot{x} = \frac{c\dot{v}}{\sqrt{vy - cc}} = \frac{c\dot{v}}{\sqrt{2av - v \times v - cc}}$, and $\dot{F}$ or $\frac{a\dot{x}}{v} = \frac{ac\dot{v}}{v\sqrt{2av - vv - cc}} = \frac{acv^{-2}\dot{v}}{\sqrt{-1 + 2av^{-1} - ccv^{-2}}}$, put z for v, to agree with the forms, then $\dot{F} = \frac{acz^{-2}\dot{z}}{\sqrt{-1 + 2az^{-1} - ccz^{-2}}}$, which comes under the 27th Form, where $p = \frac{aa}{cc} - 1 = \frac{nn}{cc}$, $v = a - \frac{cc}{z} = \frac{az - cc}{z}$, and then $\frac{ac}{cc} \times \overline{p - \frac{vv}{cc}}^{-\frac{1}{2}} \dot{v} = \frac{a\dot{v}}{c} \times \overline{\frac{nn}{cc} - \frac{vv}{cc}}^{-\frac{1}{2}} = \frac{a\dot{v}}{\sqrt{nn - vv}}$. And this comes under the 10th Form. Put z again for v; then $\dot{\varphi} = \frac{a\dot{z}}{\sqrt{nn - zz}}$; and by Form 10th, $\varphi = a \times \frac{\text{the arch}}{n}$ whoſe ſine is z, radius n. And reſtoring v, $\varphi = a \times \frac{\text{arch}}{n}$ (ſine $= v$). But becauſe Fig. 121.

Fig. 121. v was $= a - \frac{cc}{z}$, reſtore this, and $\varphi = a \times \frac{\text{arch}}{n}$. Sine $= a - \frac{cc}{z}$. And reſtoring the firſt v, $F = \frac{a}{n} \times$ arch, ſine $= a - \frac{cc}{v}$, rad. $= n$. But in A, where $F = 0$, $v = a - n$, and ſine $= a - \frac{cc}{a-n} = -f$, and the arch is a (negative) quadrant; therefore by correction, $F = \frac{a}{n} \times$ arch, whoſe coſine is $\frac{cc}{v} - a$, and rad. $= n$.

PROB. XLIV.

121. *Having given the diſtance* FN *of a planet from the upper focus* F; *to find the time of moving from the aphelion, from* A *to* N, *and the angle at* F.

Let AC $= a$, CE $= c$, CF $= n$, SN $= y$, FN $= v$. Then (by Prop. I. Sect. V. Aſtronomy,) the area PSN $= \frac{.01745caD}{2} - \frac{c}{2}\sqrt{2ay - yy - cc}$, where D $=$ degrees in the arch whoſe coſine is $\frac{a-y}{n}$, rad. $= 1$. But half the area of the ellipſis, PEA $= 180ca \times .01745$; therefore the area SAN $= \frac{180ca \times .01745}{2} - \frac{.01745caD}{2} + \frac{c}{2}\sqrt{2ay - yy - cc} = \frac{ca \times .01745}{2} \times 180 - D + \frac{c}{2}\sqrt{2ay - yy - cc} = \frac{ca \times .01745}{2} d + \frac{c}{2}\sqrt{2ay - yy - cc}$, where $d =$ degrees in the arch, whoſe coſine is $y - a$

$\frac{y-a}{n}$ or $\frac{a-v}{n}$. But area AEPB $(180ca \times .0174)$ Fig. 121.

: periodical time (t) : : area SAN $\left(\frac{ca \times .0174}{2} d + \frac{c}{2}\sqrt{2ay-yy-cc}\right)$: time of deſcribing SAN.

Or $360 : t :: d + \frac{\sqrt{2ay-yy-cc}}{a \times .0174} : d + \frac{\sqrt{2ay-yy-cc}}{.017453a} \times \frac{t}{360}$, or $d + \frac{\sqrt{2av-vv-cc}}{.017453a} \times \frac{t}{360}$ = time of deſcribing SAN.

Again, If z = cof. PSN, then (Aſtron. *ib.*) y or $2a-v = \frac{cc}{4+zn}$, from whence $z = \frac{cc-2a-v \times a}{2a-v \times n}$. And (by Trigonometry) NF (v) : S. FSN $(\sqrt{1-zz})$: : NS $(2a-v)$: S. angle SFN or AFN $= \frac{2a-v}{v}\sqrt{1-zz}$.

Hence if v be given, the time of deſcribing AN from the aphelion will be found.

And from v being given, z is found; and then the angle at the upper focus F.

Cor. *If you ſay as t the periodic time* : 360 : : *ſo the time in* SAN, *before found* : *to the angle* AFN; *then it will appear how much this value of* AFN, *differs from the true value before found.*

PROB. XLV.

Given the two ſides of the triangle ACB, AC = 16, AB = 25. *To find* CB *ſo that a body may deſcend through it, in the leaſt time poſſible.* Fig. 122.

Fig. 122. Let AB $= b$, AC $= d$, AB being parallel to the horizon, draw COD perp. to AB, $x =$ S.CAB, $y =$ its coſine. Then rad. (1) : AC (d) : : S.ACB (y) : AO $= dy$. And rad: (1) : AC (d) : : S.CAB (x) : CO $= dx$. Then CB $= \sqrt{bb + dd - 2bdy}$, let BD be perp. to BC, then CD $= \frac{CB^2}{CO} = \frac{bb+dd-2bdy}{dx}$. Now a body will deſcend through CD in the ſame time it deſcends through CB; therefore CD is a minimum, or $\frac{bb+dd-2bdy}{dx} = m$, in Fluxions, $-dx \times 2bd\dot{y} - d\dot{x} \times \overline{bb+dd-2bdy} = 0$, or $-2bdx\dot{y} = \overline{bb+dd-2bdy} \times \dot{x}$; but $xx + yy = 1$, and $x\dot{x} + y\dot{y} = 0$, and $\dot{y} = \frac{-x\dot{x}}{y}$, therefore $\frac{2bdx^2\dot{x}}{y} = \overline{bb+dd-2bdy} \times \dot{x}$, and $2bdxx = \overline{bb+dd-2bdy} \times y$, or $2bdxx = \overline{bb+dd}\sqrt{1-xx} - 2bd \times \overline{1-xx} = \overline{bb+dd}y - 2bd + 2bdxx$, and $\overline{bb+dd}.y = 2bd$, and $y = \frac{2bd}{bb+dd} =$ coſ. 24° 46′. And the angle C will be obtuſe.

PROB. XLVI.

123. *There is given the baſe* AB *of the iſoceles triangle* ADB, *and the part* DG *as far as the perp.* AG. *To find the ſides* AD, DB.

Let AD or DB $= x$, AE or EB $= b$, DE being perp. to AB, and GD $= a$. Then $AB^2 = AD^2 + DB^2 + 2GDB$, or $4bb = 2xx + 2ax$, and $xx + ax = 2bb$. And $x + \frac{1}{2}a = \sqrt{2bb + \frac{1}{4}aa}$, or $x = \sqrt{2bb + \frac{1}{4}aa} - \frac{1}{2}a$.

PROB.

PROB. XLVII.

Fig.

To find the fluent of $\frac{\dot{y}}{y} - \frac{\dot{x}}{x} = \frac{x^m\dot{x}}{ay^n}$.

Multiply by y^n, then $y^{n-1}\dot{y} - \frac{y^n\dot{x}}{x} = \frac{x^m\dot{x}}{a}$.

Aſſume the fluent $\frac{y^n}{x^r} = \frac{sx^p}{ap}$; this in Fluxions is $\frac{ny^{n-1}\dot{y}}{x^r} - \frac{ry^n\dot{x}}{x^{r+1}} = \frac{sx^{p-1}\dot{x}}{a}$, multiply this by $\frac{x^r}{n}$, then $y^{n-1}\dot{y} - \frac{ry^n\dot{x}}{nx} = \frac{sx^{p+r-1}\dot{x}}{na}$; compare this with $y^{n-1}\dot{y} - \frac{y^n\dot{x}}{x} = \frac{x^m\dot{x}}{a}$, to make them the ſame, then $\frac{r}{n} = 1$ or $r = n$, $\frac{s}{n} = 1$ or $s = n$; laſtly, $p + r - 1 = m$, and $p = m - r + 1 = m - n + 1$. Whence the fluent is $\frac{y^n}{x^n} = \frac{nx^{m-n+1}}{ap}$, or $\frac{ay^n}{x^n} = \frac{nx^{m-n+1}}{m-n+1}$.

And when x and $y = a$, then $a = \frac{na^{m-n+1}}{m-n+1}$, or $a^{m-n} = \frac{m-n+1}{n}$.

PROB. XLVIII.

If the curve RDB *be of ſuch a nature, that the tangent* DT *revolving uniformly about the curve, the particle of the curve, at any place* D, *may be as the ſquare of the ſine of the angle* ADt. *To find the nature of the curve.* 124.

Let RA $= x$, AD $= y$, RD $= z$. Then ſince DT is ſuppoſed to change its direction uniformly,

Fig. 124. therefore the angle of contact is given, and therefore z' is as the radius of curvature. But S. $<$ ADt $= \frac{\dot{x}}{\dot{z}}$; therefore $z' \propto \frac{\dot{x}\dot{x}}{\dot{z}\ddot{z}}$; whence $\frac{\dot{x}\dot{x}\dot{y}}{\dot{z}\ddot{z}}$ $\propto$ rad. curvature $\frac{\dot{z}\dot{y}}{\ddot{x}}$ ($\dot{z}$ being given). Whence (assuming the given quantity a) ($\dot{x}^2\ddot{x} = \frac{\dot{z}^2\dot{y}}{a}$, and the fluent is $\frac{\dot{x}^3}{3} = \frac{\dot{z}^2\dot{y}}{a}$, which is correct, for in R, $y = 0$, and $\dot{x} = 0$, whence $\dot{x}\sqrt[3]{\frac{a}{3}} = \dot{z}\sqrt[3]{y}$. For brevity's sake put $\frac{a}{3} = 1$, then $\dot{x} = y^{\frac{1}{3}}\dot{z}$. But $\dot{x}^2 = y^{\frac{2}{3}}\dot{z}^2 = y^{\frac{2}{3}}\dot{x}^2 + y^{\frac{2}{3}}\dot{y}^2$, and $\overline{1 - y^{\frac{2}{3}}} \times \dot{x}^2 = y^{\frac{2}{3}}\dot{y}^2$, therefore $\dot{x} = \frac{y^{\frac{1}{3}}\dot{y}}{\sqrt{1 - y^{\frac{2}{3}}}}$. Put $v = y^{\frac{1}{3}}$, and $v^3 = y$, then $\dot{x} = \frac{3v^3\dot{v}}{\sqrt{1 - vv}}$; and the fluent is $x = 2 - \frac{2 + vv}{1}\sqrt{1 - vv}$, for the nature of the curve RDB.

Reduced $4x - xx = 3v^4 + v^6 = 3y^{\frac{4}{3}} + 4yy$.

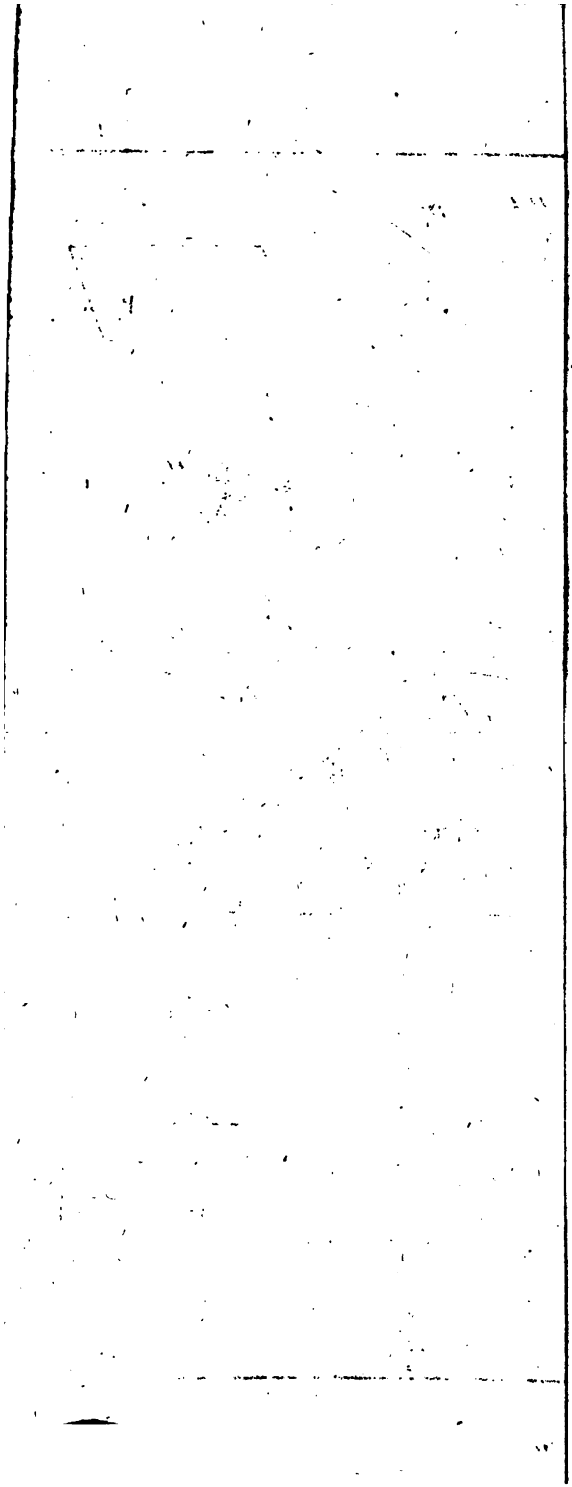
Cor. 1. *If* C *be the center of the curve, then* CR $=$ 2CB. *For when* y *is greateſt,* $\dot{x} = \dot{z}$, *and therefore* $\dot{x}$ ($\dot{z}$) $= y^{\frac{1}{3}}\dot{z}$, *and* $y^{\frac{1}{3}} = 1$, *or* $y = 1$. *Whence* $v = 1$, *and* $\sqrt{1 - vv} = 0$, *therefore* $x = 2$; *that is,* CB $= 1$, CR $= 2$.

Cor. 2. CB $= \frac{1}{3}a$.

Cor. 3. *To conſtruct the curve, take any angle, find its ſine* $\frac{\dot{x}}{\dot{z}}$, *or* $y^{\frac{1}{3}} = v$. *Then we have* $y = v^3$, *and* $x = 2 - \frac{2 + vv}{1}\sqrt{1 - vv}$; *and finding theſe for*

any

B
123
B
C
T



any ſeries of angles, a table may be made of x and y from theſe angles, and the curve conſtructed. Fig.

Otherwiſe aſſuming any value for y finds v, and v finds x.

PROB. XLIX.

Suppoſe AC, Ac, *to be curves of the ſame kind, whoſe vertices are in* A, *and their axes* AF. *To find the nature of the curve* FCE, *that ſhall cut all theſe curves* AC, Ac, *&c. at right angles.* 125.

Let two curves AC, Ac, be infinitely near, C*c* a particle of the curve FC, between them. Draw the ordinates BC, *bc*; and alſo DC, *dc*.

Let AB or DC $= x$, BC or AD $= y$, ED $= v$, AE $= d$, *o*C $= \dot{v}$, *oc* $= \dot{x}$. Draw the tangent CT to the curve CF, and the tangent CQ to the curve AC. Then the triangles C*oc*, CBQ are ſimilar, and ſubtangent QB $\left(\frac{y\dot{x}}{\dot{y}}\right)$: BC (y or $d - v$) : : C*o* ($\dot{v}$) : *oc* ($\dot{x}$), whence $\frac{y\dot{x}\dot{x}}{\dot{y}} = \overline{d - v} \times \dot{v}$, and $dv - \frac{1}{2}vv =$ fl. of $\frac{y\dot{x}^2}{\dot{y}}$.

EXAM. 1.

Let AC, Ac, be parabola's. Then BQ $= \frac{y\dot{x}}{\dot{y}} = 2x$, and $dv - \frac{1}{2}vv =$ fluent of $2x\dot{x} = x\dot{x}$, or $xx = \frac{1}{2} \times \overline{2dv - vv}$, an equation to an ellipſis, therefore FCE is an ellipſis. And if AE $= 10$, then AF $= \frac{10}{\sqrt{2}} = 7$ nearly.

EXAM. 2.

Let AC, Ac, be ellipſes, whoſe tranſverſe is given $= 2a$, then $2ax - xx = byy$, and $a\dot{x} - x\dot{x} =$

Fig. 125. $= byy$, whence $\frac{yx\dot{x}\dot{x}}{\dot{y}} = \frac{yx\dot{x}\dot{x} \times by}{a\dot{x} - x\dot{x}} = \frac{byyx\dot{x}}{a-x} = \frac{2ax - xx}{a-x}\dot{x} = \frac{aa}{a-x}\dot{x} - a\dot{x} + x\dot{x}$, and the fluent is $dv - \frac{1}{2}vv = 2.3025aa \times \log. \frac{a}{a-x} - ax + \frac{1}{2}xx$, for the equation of the curve.

In the hyperbola $\frac{yx\dot{x}\dot{x}}{\dot{y}} = \frac{2ax + xx}{a+x}\dot{x} = a\dot{x} + x\dot{x} - \frac{aa\dot{x}}{a+x}$, and the fluent, $dv - \frac{vv}{2} = ax + \frac{1}{2}xx - 2.3025aa \times \log. \frac{a+x}{a}$.

PROB. L.

To find the sum of the series $z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7}$ &c.

Here some value must be put for z, as suppose 1. Then the series is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$ &c. that is, (collecting two terms into one) $\frac{2}{1.3} + \frac{2}{5.7} + \frac{2}{9.11}$ &c. or $2 \times$ into $\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11}$ &c. But by Prob. XXVIII. Increments, the sum of the series $\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11}$ &c. $= T + \frac{1}{4v} + \frac{4}{4v}A + \frac{2.3.4}{6.v^2}B +$ &c. $= T + \frac{1}{4.41} + \frac{4}{4.45}A + \frac{2.3.4}{6.49}B + \frac{3.5.4}{8 \times 53}C$ &c. where $T =$ the

the sum of 10 terms $= .38644$, $v = 41$, $v = 45$, $v = 49$, &c. and the whole sum $= .39270$, and $2 \times .39270 = .7854$, the sum of the given series. Fig. 125.

PROB. LI.

To find the sum of the series $2\sqrt{d} \times$ *into* $1 + \frac{1.1}{2.5} A + \frac{3.5}{4.9} B + \frac{5.9}{6.13} C + \frac{7.13}{8.17} D$, *&c. being the fluent of* $\overline{dd - xx}^{-\frac{1}{2}} \times dx^{-\frac{1}{2}} \dot{x}$, *when* $x = d$; *which is the time of descent of a body in a quadrant, whose radius is d.*

Here the z^{th} term is $\frac{2z-3}{2z-2} \times \frac{4z-7}{4z-3} \cdot s = \frac{z-\frac{3}{2}}{z-1} \times \frac{z-\frac{7}{4}}{z-\frac{3}{4}} \cdot s = s$; put $y = z - 1$, or $z = y + 1$, then $z = y = 1$. And the equation is now $\frac{y-\frac{1}{2}}{y} \times \frac{y-\frac{3}{4}}{y+\frac{1}{4}} \cdot s = \frac{y-\frac{1}{2}}{y} \times \frac{y-\frac{3}{4}}{y-\frac{3}{4}} \cdot s$, s being the $z+1^{th}$ term; according to the directions in Prop. XV. Increments.

To find the integral, compare this with Ex. 26. Prop. XIII. Increments, and we have $m = \frac{1}{2}$, $n = \frac{3}{4}$, $z = y = 1$. And we shall have the integral $\overline{z-n}, \cdot s \times$ into $- \frac{1}{m} + \frac{\frac{n}{z} P}{m+1} +$ &c. Now this series being negative, does not give the sum of the series to the z^{th} term, but the sum of all the terms, *ad infinitum*, beyond the z^{th} term. For the greater z is, the less the sum will be. Therefore sum up 4 terms of the series $= 1.16570$, and the 5^{th} term will be .01608; and writing y for z, we shall have for 4 terms 1.16570, and the rest *ad infinitum*

Fig. 125. $finitum = \overline{y-n} \times s \times \frac{1}{m} + \frac{\frac{n}{y}P}{m+1} + \frac{\frac{n+1}{y+1}Q}{m+2} + \frac{\frac{n+2}{y+2}R}{m+3}$ &c. that is (putting $y=5$) $4\frac{1}{4} \times .01608 \times \frac{1}{0\frac{1}{2}} + \frac{\frac{3}{4.5}P}{1\frac{1}{2}} + \frac{\frac{7}{4.6}Q}{2\frac{1}{2}} + \frac{\frac{11}{4.7}R}{3\frac{1}{2}} + \frac{\frac{15}{4.8}S}{4\frac{1}{2}}$ &c.

P =	.06834	2P =	.13668
Q =	.01025	$\frac{2}{3}$Q =	683
R =	299	$\frac{2}{5}$R =	119
S =	117	$\frac{2}{7}$S =	33
T =	55	$\frac{2}{9}$T =	12
U =	29	$\frac{2}{11}$U =	5
X =	17	$\frac{2}{13}$X =	3
Y =	10	$\frac{2}{15}$Y =	2
		sum =	.14525

series	.14525
add the first 4 terms	1.16570
total sum	1.31095
multiply by	$2\sqrt{d}$
	$2.62190\sqrt{d}$ for the sum of all.

Otherwise.

It may likewise be solved by *Sterling*'s Series, Prop. VIII. Here the equation of the series is $T' = \frac{z-\frac{1}{2}}{z} \times \frac{z-\frac{3}{4}}{z+\frac{1}{4}} T$, and the successive values of z are 1, 2, 3, 4, &c. and $m = \frac{1}{4}$, $n = \frac{3}{4}$, sum up the 4 first terms of the series $= 1.16570$, and the fifth term or $T = .01609$, and $z = 5$. Then $A = 4\frac{1}{4}T$, $B = \frac{3}{4.5} A$, $C = \frac{7}{4.6} B$, $D =$

$\frac{11}{4.7}$ C, E $=$ $\frac{15}{4.8}$ D, &c. and S $=$ 2A $+$ $\frac{2}{3}$B $+$ $\frac{3}{5}$C $+$ $\frac{2}{7}$D $+$ $\frac{3}{9}$E, &c. which brings out the ſame anſwer as before. Fig.

PROB. LII.

Suppoſe the bob of a pendulum ABC *to be in the form of a paraboloid, whoſe axis is* BD. *To find the center of oſcillation.* 126.

Let BD $= x$, AD $= y$, $2rx = yy$. And ſuppoſe BD parallel to the axis of motion XY. Alſo let $c = 3.1416$, D$f = z$. Then (by Prob. XVIII. Sect. II. Fluxions,) in the circle DA the fluxion of the figure is $2c \times Df \times Df^2 = 2cx^3\dot{x}$, and the fluent $= \frac{1}{2}cx^4 = \frac{1}{2}cy^4$, at laſt, $= \frac{1}{2}c \times 4rrxx$, and $\dot{G} = 2crrxx\dot{x}$, and G $= \frac{2}{3}crrx^3$. But the ſolidity of the paraboloid ABC $= \frac{cyyx}{2} = B = crxx$; therefore $\frac{G}{dB} = \frac{2crrx^3}{3crxxd} = \frac{2}{3} \times \frac{rx}{d} = \frac{yy}{3d}$. And the diſtance from the axis of motion $= d + \frac{1}{3} \times \frac{yy}{d}$.

PROB. LIII.

To find the denſity of the ſun, compared with that of the earth.

Let D $=$ denſity of the ſun, $d =$ denſity of the earth, P $=$ earth's parallax as ſeen from the ſun $=$ ſun's apparent diameter ſeen from the earth, $p =$ moon's mean horizontal parallax, T $=$ a year, or the earth's periodical time, $t =$ moon's periodical time. Then (by Prop. XIX. Centripetal Forces) the denſities of central bodies are reciprocally as the cubes of the parallaxes, and ſquares of the

Fig. 126. the periodical times; that is, $D : d :: \frac{1}{P^3T^2} : \frac{1}{p^3t^2} :: p^3tt : P^3T^2$, and $D = \frac{p^3ttd}{P^3T^2}$.

Cor. *Hence the denſity of the ſun is had, without either the ſun's diſtance or his parallax.*

PROB. LIV.

The latitude of the place being given; to find the time of ſhorteſt twilight.

It is found by obſervation, that the time of day-break is when the ſun is 18 degrees below the horizon.

127. Let ZHNO be the meridian, HO the horizon, EQ the equinoctial, BD the parallel of day-break, P*p* the earth's axis, Z the zenith, N the nadir, *os* the ſun's parallel of declination when twilight is ſhorteſt. Draw the parallel *rv* infinitely near *os*, and draw the hour circles *op*, *rap*, *tsp*, *vp*. Now when the portion of the parallel *os*, comprehended between HO and BD is the ſhorteſt, it will be equal to the portion *rv*; becauſe when it is the leaſt, it neither increaſes nor decreaſes; therefore *oa* the decreaſe at *o*, is equal to *tv* the increaſe at *s*. But in the triangles *oar*, *tsv*, *ar* = *st*, and *oa* = *tv*, and the angles at *a* and *t* are right, therefore the angle *roa* = angle *tvs*. But by Trigonometry the angle *po*N of the ſpherical triangle *po*N is = *roa*. And the angle *pv*N, of the ſpherical triangle *pv*N is = *tvs*. Therefore to find the angles of theſe ſpherical triangles. Let rad. = 1, *x* = *s.* declination = cof. *po*, or *pv*, *y* = *s. po*, or *pv*, *l* = *s.* latitude = cof. *p*N, *s* = *s.* 18° = cof. N*v*, *c* = *s.* N*v*. Then (by Caſe 11th, Spherical Trigonometry) cof. *po*N (or *roa*) = $\frac{l}{1 \times y}$, becauſe the coſine of N*o* (being

(being a quadrant) is *o*. Alſo the coſ. *pv*N (or *tvs*) Fig. $= \frac{l - sx}{cy}$; therefore $\frac{l}{y} = \frac{l - sx}{cy}$, and $cl = l - sx$, 127.

therefore $sx = l - cl$, and $x = \frac{1 - c}{s} l$; but $1 - c$ is the verſed ſine of 18°. Therefore as *s.* 18° : verſ. 18° l : *s.* latitude : *s.* declination of the ſun ſouth, when twilight is the ſhorteſt.

Cor. *Hence, as radius : tan.* 9° : : *s. latitude : s. ſun's declination ſouth, when twilight is ſhorteſt.*

For (by Cor. 1, Prop. II. Trigonometry) radius : tangent of an arch : : ſine of 2 arch : verſ. twice the arch; that is, rad. : tan. 9° : : *s.* 18° : verſ. 18° : : *s.* latitude : *s.* declination.

It is ſaid J. Bernoulli was 5 years in ſolving this Problem, and did it but very audwardly at laſt.

PROB. LV.

To find the proportional heat of the ſun in all latitudes, for any day.

The action of the ſun's rays is, as all other impulſes or ſtrokes, more or leſs forcible, according to the ſines of the angles of incidence; whence the vertical ray (giving the greateſt heat) being put equal to radius, the force of any other ray, upon the horizon, will always be as the ſine of the ſun's altitude. This being granted, it will follow, that the time of continuance of the ſun's ſhining being taken for a *baſis*, and the ſines of the ſun's altitudes erected thereon as perpendiculars, and a curve drawn through their extremities; the area of that curve will be proportional to the heat of the ſun in that time.

Let AB be the length of a winter day, EQ the 128.
length of an equinoctial day, CD the length of a ſummer day. Or we may ſuppoſe CD the diurnal arch,

Fig. 128. arch, reckoned on the equinoctial, to repreſent the time.

Let l, m = ſine and coſine of latitude; d, c = the ſine and coſine of the ſun's declination; z = PF the arch from noon, y, x = ſine and coſine of PF; rad. = 1. Then by Aſtronomy, rad. : tan. latitude : : tan. declination : *s.* aſcenſional difference EA (or EC); then AF the ſemidiurnal arch is known = EF — EA, (or EF + EC) let n = *s.* AF, or CF.

Then GF is the ſine of the meridian altitude. And to find MP the ſine of the altitude, at the diſtance of time FP from noon, we have two ſides and the included angle of a ſpherical triangle, to find the third ſide, therefore (by Caſe 8*th*, Spherical Trigonometry) $mcx \pm ld$ = coſ. zenith diſtance = *s.* altitude MP. Then MP × $\dot{z}$ or $mcx\dot{z} \pm ld\dot{z}$ = flux. of the area GFPM. But by the nature of the circle $x\dot{z} = r\dot{y} = \dot{y}$; whence the flux. area = $my \pm ld\dot{z}$, and $my \pm ldz$ = area FGMP. And $mcn \pm ld \times$ AF = area AGF; and $mc \times \overline{n - y} \pm ld \times$ AP = area AMP, the heat generated in the time AP, uſing + for ſummer, and — for winter.

Hence the heat generated in the whole day is = $2mcn + ld \times$ CD, for ſummer; and $2mcn - ld \times$ AB for winter.

Cor. 1. *Under the equinoctial the heat is* $2crr$; *and when the ſun has no declination, the heat is* $2r^1$.

Cor. 2. *Under the pole the heat of a ſummer day is* $rd \times 2$EQ.

Cor. 3. *Under the frigid zones, where the ſun ſets not, the heat is* $ld \times 2$EQ; *and therefore is* 2EQ × *fine of the ſun's altitude at* 6.

Cor. 4. *In any latitude when the ſun is in the equinoctial, the heat is* $2mrr$.

Cor. 5. *The greateſt heat under the equinoctial, to the greateſt heat under the poles; is as* $2r^1$ *to* $rd \times$ 2AE,

2AE, *or as r to d* × 3.1416; *or as r to* .3987 × 3.1416; *that is, as 4 to 5.* Fig. 128.

Cor. 6. *In the latitude of London, the heat of the longeſt ſummer's day, to that of the ſhorteſt winter day, is as* $6\frac{1}{2}$ *to* 1.

SCHOL.

Dr. Halley, in ſolving this Problem, remarks, that ſince the nature of heat is to remain in the ſubject, after the cauſe that heated it is removed; and particularly in the air under the equinoctial, the 12 hours abſence of the ſun, does very little ſtill the motion impreſſed, by the paſt action of the rays wherein heat conſiſts, before he ariſe again. But under the pole, the long abſence of the ſun for 6 months, wherein the extremity of cold does obtain, has ſo chilled the air, that it is as it were frozen, and cannot, before the ſun has got far towards it, be any way ſenſible of his preſence; his beams being obſtructed by the thick clouds, and perpetual fogs and miſts, and by that *atmoſphere of cold*, as the late honourable Mr. *Boyle* was pleaſed to term it, proceeding from the everlaſting *ice*, which in immenſe quantities does chill the neighbouring air, and which the too ſoon retreat of the ſun leaves unthawed, to encreaſe again during the long winter that follows this ſhort interval of ſummer.

But the different degrees of heat and cold in different places, depend in a great meaſure upon the accidents of the neighbourhood of high mountains, whoſe height exceedingly chills the air brought by the winds over them; and of the nature of the ſoil, which variouſly retains the heat; particularly the ſandy, which in *Africa*, *Arabia*, and generally where ſuch ſandy deſarts are found, do make the heat of the ſummer incredible to thoſe that have not felt it.

To which I may add, that the ſine of the angle of incidence, or obliquity of the rays, is not the ſole

Fig. 128. ſole cauſe of the proportion of heat. But the quantity of atmoſphere through which the light paſſes, has got a great effect upon it; for the more atmoſphere it comes through the more light will be deſtroyed, and conſequently the heat diminiſhed upon that account, though the angle of incidence was to remain the ſame. Therefore it will follow that the quantity of heat will be leſs, when the ſun is lower than in the ratio of the ſine of incidence, as ſuppoſed in this calculation. See Prop. XVIII. B. I. Optics.

PROB. LVI.

To find the proportional heat of the ſun for a year in any latitude, as London.

Take a ſummer and a winter day where the ſun's declination is the ſame; then by the laſt Prob. the heat of the

128. ſummer day $= 2mcn + ld \times CD$
winter day $= 2mcn - ld \times AB$
ſum $= 4mcn + ld \times 4AE$, or taking a quarter thereof, the heat for theſe two days is $= mc \times$ cof. AE $+ ld \times$ AE, where AE is the aſcenſional difference. And if this be done every day from the equinox to the ſolſtice, the ſum of all is the heat, for half a year.

Now to approximate to the ſum of theſe, find the aſcenſional difference for every 10 or 15 degrees of the ecliptic thus; let $t =$ tan. latitude, $s =$ *s.* ſun's longitude, $p = s.23\frac{1}{2}$ the obliquity of the ecliptic; then rad. (1) : s : : p : $ps = d$ the *s.* declination, and rad. (1) : t : : tan. decl. $\left(\frac{ps}{c}\right)$: $\frac{pts}{c}$: *s.* aſcen. difference, which gives AE and its coſine. Thus for 0, 15, 30, 45, &c. degrees of the ecliptic, the declination, and aſcenſ. difference, and the

the other requisites will be as in the following Table. Fig. 128.

⊙ longitude.	declination.	ascen. differ.	cos. decl. × cos. asc.	S.decl. × asc.
0	° ′	0	1.0000	0.
15	5 55	7.483	.9862	.7723
30	11 30	14.816	.9473	2.9537
45	16 23	21.68	.8915	6.1128
60	20 12	27.55	.8321	9.5138
75	22 39	31.63	.7857	12.1827
90	23 30	33.13	.7505	13.2107

Then by the rule for 7 ordinates (Cor. 1. Prop. IX. Differential Method) we have,

$$\left.\begin{array}{l} A = 1.7505 \\ B = 1.7719 \\ C = 1.7794 \\ D = .8915 \end{array}\right\} \times m, \text{ and } \left.\begin{array}{l} A = 13.2107 \\ B = 12.9550 \\ C = 12.4675 \\ D = 6.1128 \end{array}\right\} \times l \times .01745$$

And the sum of the first series = 50.14 } where L =
And sum of the second series = 7.99 } 91 days.

total 58.13, and this ×8,

for the reasons before given, gives 465.04 the heat in a whole year in lat. $51°\frac{1}{2}$.

Cor. 1. *Under the equinoctial,* $l = 0$, $m = r$, *and cos.* AE $= r$. *Therefore only the sum of all the crr must be calculated.*

⊙ lon.	⊙decl.	cos. decl.
0	0° ′	1.0000
15	5 55	.9946
30	11 30	.9799
45	16 23	.9594
60	20 12	.9385
75	22 39	.9229
90	23 30	.9170

Fig. 188.

Then $\left.\begin{array}{l} A = 1.9170 \\ B = 1.9175 \\ C = 1.9184 \\ D = .9594 \end{array}\right\} \times rr$ or 1.

Then by the foreſaid rule, the ſum of all is 88.48, which × by 8, gives 697.8 for the equinoctial heat in a year. And hence the heat under the equinoctial, to that at London, is as 3 to 2.

Cor. 2. *Under the poles,* $m = 0$, *and* $l = r$, *and* AE *muſt be ſuppoſed a quadrant* $= q = \frac{3.1416}{2}$. *Therefore we muſt find the ſum of all the rqd.*

⊙ lon.	⊙decl.	S. declin.
0	0	0
15	5 55	.1031
30	11 30	.1993
45	16 23	.2821
60	20 12	.3453
75	22 39	.3851
90	23 30	.3987

Then $\left.\begin{array}{l} A = .3987 \\ B = .4882 \\ C = .5446 \\ D = .2821 \end{array}\right\} \times rq \times 8.$

And by the ſaid Rule the ſum of all $= 67.375 \times 4 = 269.5$, for the polar heat in a year; or if you will in half a year; for in the other half year, it gets none. And therefore the heat at London, to that under the Pole, is nearly as 12 to 7.

Cor. 3. *And hence the equinoctial heat is to the polar heat, as* 13 *to* 5 *nearly.*

PROB. LVII.

In the ſtereographic projection of the ſphere, to draw a leſſer circle perpendicular to the plane of projection, which lies very near its parallel great circle.

When

When the ſcheme is large, as in the drawing of maps, and the circle lies very near its primitive great circle, it is not practicable to draw ſuch a circle with a pair of compaſſes, as directed by the rules of that projection. And therefore it muſt be done after ſome other manner, as follows. Fig.

1 *Way.*

Let ADBE be the primitive, DCE the parallel great circle, GHI the circle to be drawn. Draw GN, I*n* perp. to DE. It is plain by the nature of this projection, that NG or *n*I is the ſine of DG, the circle's diſtance from its parallel great circle DE; alſo CH is the tangent of half DG; and theſe being ſet off from DE will give the three points G, H, I, through which deſcribe the portion of the circle GHI with a bow, as directed in the Scholium to Prop. XII. of the Projection. 129.

2 *Way.*

But when more exactneſs is required, ſeveral points muſt be found, through which to draw the circle.

Let s = nat. ſine, t = tangent, f = ſecant of the arch GA, rad. = 1. Then rad. FH = t, FC = f, by Prop. V. Projection. And in the triangle FGC, FG (t) : S.GA or $\angle$ C (s) : : GC (1) : S.GFC or arch HG in degrees. Then let z = ſine, v = verſed ſine, of any arch Hg leſs than HG; then tz = ſine, and vz = verſed ſine of Hg to the radius t, let m = tan. of half DG. Then make Cl = tz, and perp. thereto make $lg = m + tv$, and g will be in the curve. And thus find as many points g as you will, and draw a curve line thro' them.

3 *Way.*

Draw the tangent HO, and put Hp = n, $pg = x$, and diameter 2FH = 2t, then by the property of the circle $Hp^2 = pg \times \overline{2FH - pg}$, that is, $nn = x \times \overline{2t - x}$, or $2tx - xx = nn$, and $tt - 2tx + xx = tt - nn$;

Fig. 129. $tt - nn$; whence $x = t - \sqrt{tt - nn} = \frac{nn}{2t} + \frac{n^4}{8t^3}$ $= pg$, and g is in the curve; in moſt caſes $\frac{n^4}{8t^3}$ may be left out. And thus any number of points may be found, thro' which the curve muſt be drawn.

4 Way.

130. This may be performed mechanically thus. Having found as before, the 3 points G, H, I, take a thin plane, as a ſtrong paper or paſteboard, PQR, whoſe edge PR is very ſtreight, lay that edge to the three points IHD, and (ſuppoſing the lines DH, HG, drawn) make the angle PST = DHG, and cut off the part PST. Then apply the ſide ST (of the inſtrument TSRQ) to the point G, and the ſide SR to the point I, and where the angle S falls, make a point g. Then ſhifting the inſtrument, ſtill keeping the points G, I, in the ſides ST, SR; the angle S will find another point g; and by this means you may find as many points as you will, through which to draw the curve.

Note, the curve may be drawn by a continued motion, thus. Stick up two pins at G and I, and move the inſtrument about, ſo that the ſide ST may continually ſlide by the pin G, and the ſide SR by the pin I; then the point S will deſcribe the curve required.

By this method there will be required a new inſtrument to be made for every parallel. And this method will deſcribe any parallel, whether near to the parallel great circle, or far off.

PROB. LVIII.

131. *Having given the curve* ZFO, *by the revolution of which about its axis* ZH, *a dome or cupolo is formed. To find the exterior curve* SGg, *ſo that all the arch ſtones*

ſtones lying aſlope, the joints may be perpendicular to the inner curve ZF; *and the whole may ſtand in equilibrio.* Fig. 131.

Let HZFO, HZL*o*, be two plains intercepting a portion of the cupolo, which is conſtructed upon this ſuppoſition, that the arch ſtones, as IF, run quite through the thickneſs of the arch; in that poſition, and may freely ſlide up or down upon the joint.

1. When the joints meet in one point H, the curve ZF being a circle. Suppoſe the plane NDB drawn parallel to the horizon. The weight of the arch ſtone FG*tf*, by its weight tends in a line parallel to ZH; and the preſſures againſt the joints FH, *f*H, are perpendicular to theſe joints. Therefore theſe three forces, the weight and two preſſures, will be as three lines drawn perpendicular to their directions; that is, as D*d*, DH, *d*H. And this would be ſo every where, if the depth FL was every where the ſame, but it grows leſs towards Z; in proportion to FM, or DB, or *nd*, which is the depth of the pyramid HF*l* at *d*. Therefore if the curve SG*g* be ſuch, that the ſolid or arch ſtone FG*rtf* be always equal to the correſpondent pyramid HD*dn*; then G will be in the curve required. But the three pyramids HD*n*, HF*l*, and HG*t*, are as HD^3, HF^3, and HG^3; whence $HD^3 : HG^3 - HF^3$: : pyramid HD*n* : ſolid FG*rtf*. Therefore if $HD^3 = HG^3 - HF^3$ (or $HG = \sqrt[3]{HD^3 + HF^3}$, then the pyramid HD*n* = arch ſtone F*t*. Conſequently taking $FG = \sqrt[3]{HD^3 + HF^3} - HF$, then G will be in the curve; and if $ZS = \sqrt[3]{HB^3 + HZ^3} - HZ$, then S is in the curve. Or if the point S be given, then taking $HB = \sqrt[3]{HS^3 - HZ^3}$, then the point B is given, thro' which BN is to be drawn parallel to the horizon, by which the points G, *g*, &c. will be found.

2. Let ZFO be any curve, FH the radius of curvature at F; draw BN parallel to the horizon, 132.

and

Fig. 132. and PD, P*d*, P*n*, parallel to HF, H*f*, H*l*. And if D*d* expreſs the weight of the arch ſtone FG*rtf*, then PD, P*d*, will be the preſſure of the joints FH, *f*H. Therefore by reaſoning as before, if the pyramid PD*n* be equal to the arch ſtone FG*tf*, then G will be in the curve. Whence if $PD^3 = HG^3 - HF^3$, or $HG = \sqrt[3]{PD^3 + HF^3}$, then G is in the curve. Therefore if the point S be given, take $PB = \sqrt[3]{HS^3 - HZ^3}$, HZ being the radius of curvature in Z. Then BN being drawn parallel to the horizon; from this all the points G will be found, as before mentioned.

EXAM. 1.

133. Let ZO be a circle, HZ = 1, $ZS = \frac{1}{10}$, then HB = .69; then if ZF = 30 deg. then FG = .14. If ZF = 60 deg. then FG = .55, &c. for deſcribing the curve SGG.

EXAM. 2.

134. Let ZF*f* be a parabola, ZA = AO = 1, $ZS = \frac{1}{8}$, M the focus, $ZM = \frac{1}{4}$, $Zm = \frac{1}{2}$, rad. curvature at $Z = \frac{1}{2}$, at F = 1.41 = FH, at *f* = 2.6 = *fb*. Then (PB or) $AB = \sqrt[3]{\frac{61}{512}} = .49$; then $HG = \sqrt[3]{AD^3 + HF^3} = 1.47$, and FG = .06; and $bg = \sqrt[3]{Ad^3 + bf^3} = 2.63$, and $fg = .03$; through which points, the curve SG*g* is to be deſcribed.

PROB. LIX.

To find the fluent of $cx^2\dot{x} + y\dot{x} = a\dot{y}$.

This reduced is $\dot{y} = \frac{cx^2\dot{x}}{a} + \frac{y\dot{x}}{a}$, and the fluent is eaſily found by *Sir J. Newton*'s method (Prob. II. Caſe II.) as follows.

$cx^2\dot{x}$

Fig. 134.

$$\begin{array}{l|l} & \frac{cx^2\dot{x}}{a} \\ \hline +\frac{y\dot{x}}{a} & +\frac{cx^3\dot{x}}{3aa}+\frac{cx^4\dot{x}}{4aa}+\frac{cx^5\dot{x}}{4\cdot 5a^3} \\ \hline \dot{y} & =\frac{cx^2\dot{x}}{a}+\frac{cx^3\dot{x}}{3aa}+\&c. \\ y & =\frac{cx^3}{3a}+\frac{cx^4}{3\cdot 4aa}+\frac{cx^5}{3\cdot 4\cdot 5aa}+\frac{cx^6}{3\cdot 4\cdot 5\cdot 6a^3}\&c. \end{array}$$

$$y = 2caa\times\frac{x^3}{2.3a^3}+\frac{x^4}{2.3.4a^4}+\frac{x^5}{2.3.4.5a^5}\&c.$$

$$=2caa\times 1+\frac{x}{a}+\frac{x^2}{2aa}+\frac{x^3}{2.3a^3}+\frac{x^4}{2.3.4a^4}+\frac{x^5}{2.3.4.5a^5}\&c.$$

$$-2caa\times:1+\frac{x}{a}+\frac{xx}{2aa}$$

$$=2caa\times\text{numb. of hyp. log}\ \frac{x}{a}-2caa-2cax-cxx$$

$$=2caa\times M^{\frac{x}{a}}-2caa-2cax-cx^2.$$

Where M = 2.71828, the number whoſe hpy. log. is 1.

PROB. LX.

To find the line of the firſt ſatellite of Jupiter's moving through the penumbra of Jupiter.

The apparent diameter of the ſun is $32'\ 12''$ at the earth, and at Jupiter it is reciprocally as the diſtance. But Jupiter's mean diſtance is to the earth's mean diſtance from the ſun, as 5 2 to 1. Therefore as 5.2 : 1 : : $32'\ 12''$ or $1932'' : 371.5''$; and this is the angle that the penumbra makes at Jupiter.

The periodic time of the firſt ſatellite is $1^d\ 18^h\ 27'\ 34''$ and that is = 152854 ſeconds. And the angle of the penumbra being 371.5 ſeconds. There-

Fig. 134. Therefore as 360 deg. or 1296000″ the whole circumference : to 152854″ the whole time of revolution : : ſo is 371″.5 the angle of the penumbra : to 44″ of time nearly. And therefore all the time the ſatellite ſtays in the penumbra, amounts only to 44 ſeconds of time.

Cor. *The difference of time, between the ſatellite's appearing after its emerging out of Jupiter's ſhadow, when obſerved by the beſt and the worſt teleſcopes, can never amount to* 44 *ſeconds; and very probably not to half as much.*

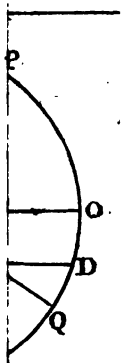
This matter is eaſily tried with two teleſcopes of different goodneſs. But there may be as much difference in the goodneſs of eyes, as in the goodneſs of teleſcopes; and reaſon tells us not to apply bad ones of either ſort.

FINIS.

ERRATA.

In page 430,

	instead of		read	
0	p		p	
1	$rA - 1$	$= A$	$rp - 1$	$= A$
2	$rB - 2$	$= B$	$rA - 2$	$= B$



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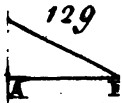

the end

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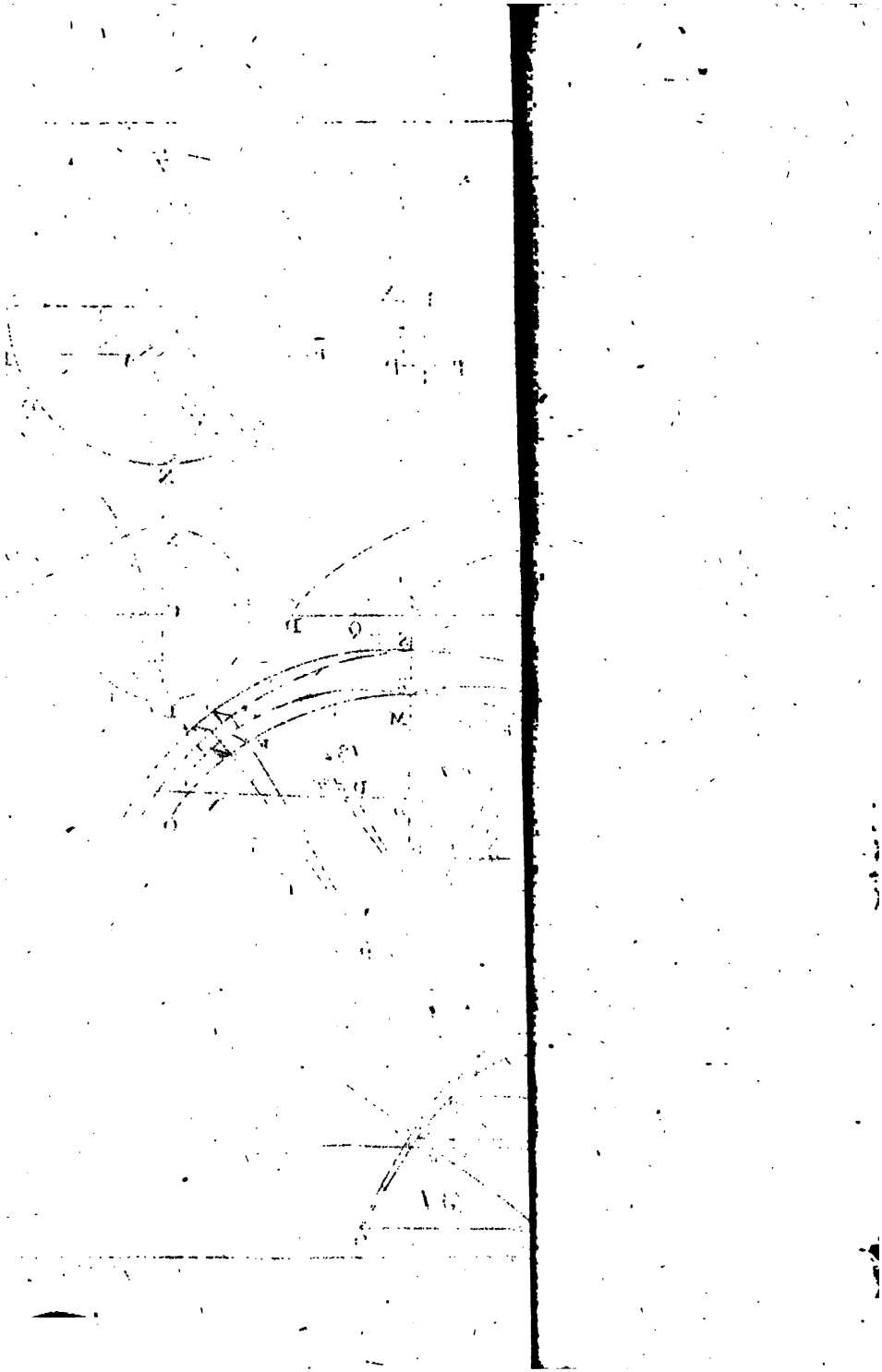

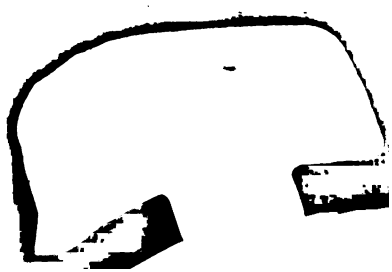

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